



UNIVERSITAS GADJAH MADA  
FAKULTAS TEKNIK  
DEPARTEMEN TEKNIK SIPIL DAN LINGKUNGAN

# SISTEM PERSAMAAN LINEAR

*Systems of Linear Algebraic Equations*

# Sistem Persamaan Linear

## ❑ Acuan

- ❑ Chapra, S.C., Canale R.P., 1990, *Numerical Methods for Engineers*, 2nd Ed., McGraw-Hill Book Co., New York.
  - Chapter 7, 8, dan 9, hlm. 201-290.

# Sistem Persamaan Linear

- Serangkaian  $n$  persamaan linear

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = c_1$$


$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = c_2$$

.

.

.

$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = c_n$$



Sejumlah  $n$  persamaan linear ini harus diselesaikan secara simultan untuk mendapatkan  $x_1, x_2, \dots, x_n$  yang memenuhi tiap persamaan dalam sistem persamaan ini.

# Metode Penyelesaian

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## Jumlah pers. sedikit, $n \ll$

- Penyelesaian
  - Grafis
  - Cramer
  - Eliminasi

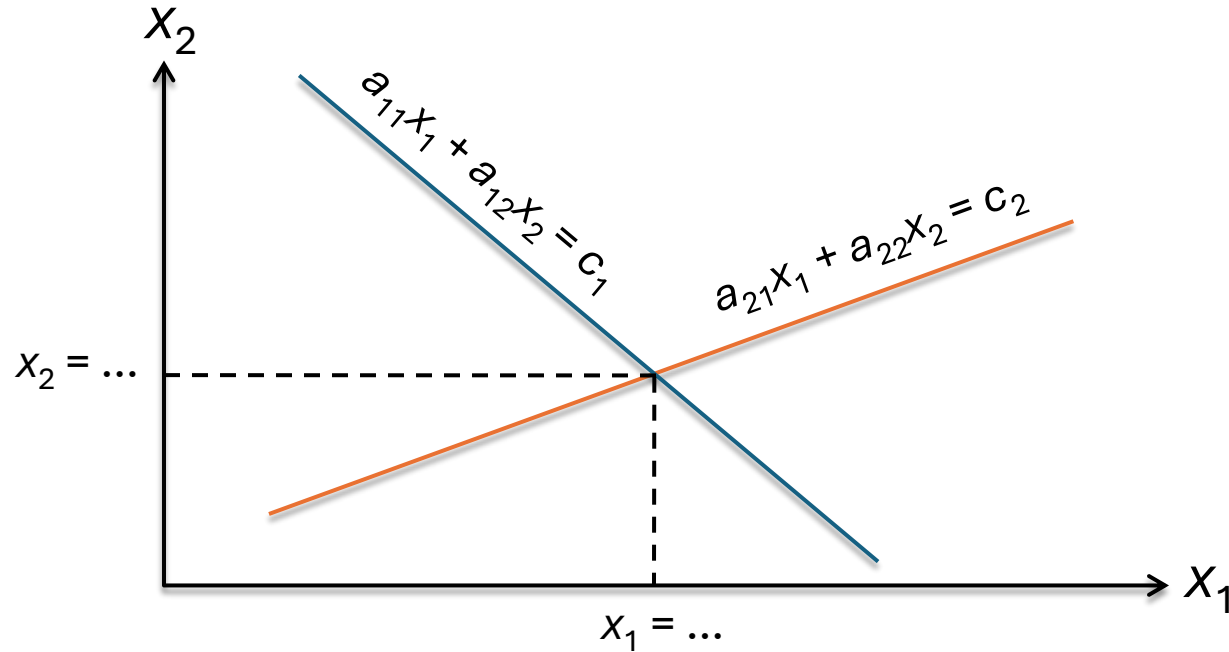
## Jumlah pers. banyak, $n \gg$

- Penyelesaian langsung
  - Eliminasi Gauss
  - Gauss-Jordan
  - Dekomposisi  $LU$
  - Inversi matriks
- Iteratif
  - Jacobi
  - Gauss-Seidel
  - *Successive Over Relaxation*

# Metode Grafis

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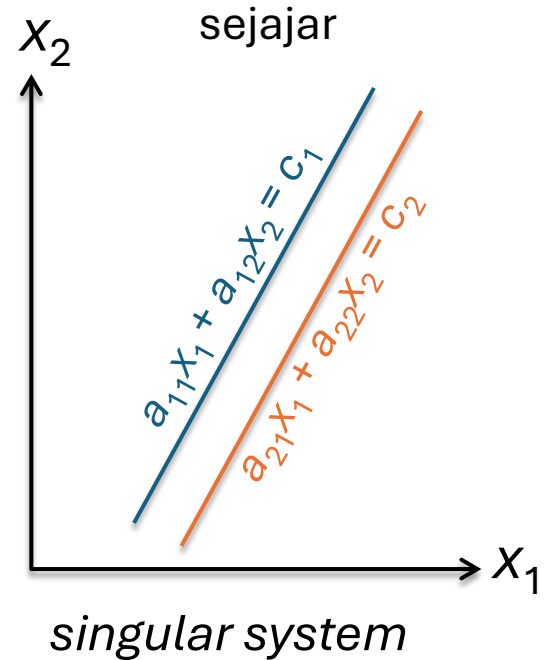
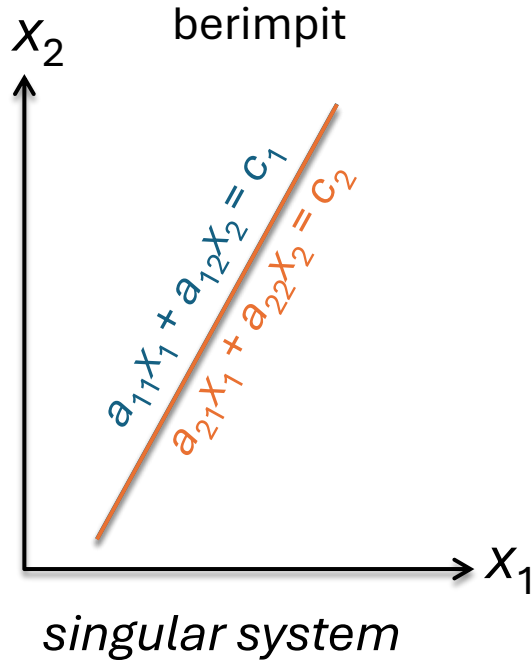
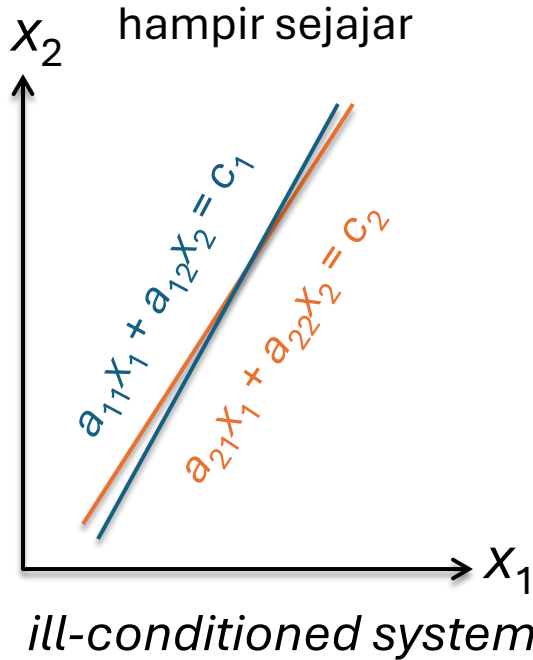
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# Metode Grafis

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# Metode Cramer

- ❑ Variabel tak diketahui,  $x_i$ , merupakan perbandingan dua determinan matriks
  - ❑ **Penyebut** determinan,  $D$ , matriks koefisien sistem persamaan
  - ❑ **Pembilang** determinan matriks koefisien sistem persamaan seperti penyebut, namun koefisien kolom ke- $i$  diganti dengan koefisien  $c_i$
- ❑ Contoh
  - ❑ 3 persamaan linear
$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = c_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = c_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = c_3$$

# Metode Cramer

$$[A] = \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$D = \det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$x_1 = \frac{\begin{vmatrix} c_1 & a_{12} & a_{13} \\ c_2 & a_{22} & a_{23} \\ c_3 & a_{32} & a_{33} \end{vmatrix}}{D}$$

$$x_2 = \frac{\begin{vmatrix} a_{11} & c_1 & a_{13} \\ a_{21} & c_2 & a_{23} \\ a_{31} & c_3 & a_{33} \end{vmatrix}}{D}$$

$$x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & c_1 \\ a_{21} & a_{22} & c_2 \\ a_{31} & a_{32} & c_3 \end{vmatrix}}{D}$$

# Determinan Matriks

- ❑ Matriks bujur sangkar  $n \times n$
- ❑ Mencari determinan matriks
  - ❑ Hitungan manual
  - ❑ MSExcel, dengan fungsi **=MDETERM()**
- ❑ Contoh hitungan determinan matriks  $2 \times 2$  dan  $3 \times 3$

$$[A] = \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad [B] = \mathbf{B} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

# Determinan Matriks

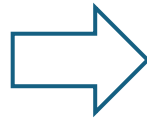
$$D = \det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\begin{aligned} D = \det \mathbf{B} &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \end{aligned}$$

# Metode Cramer

- Contoh: 3 persamaan linear

$$\begin{aligned}3x_1 - 0.1x_2 - 0.2x_3 &= 7.85 \\0.1x_1 + 7x_2 - 0.3x_3 &= -19.3 \\0.3x_1 - 0.2x_2 + 10x_3 &= 71.4\end{aligned}$$



$$\mathbf{A} \cdot \mathbf{X} = \mathbf{C}$$

$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0.1 & 7 & -0.3 \\ 0.3 & -0.2 & 10 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.3 \\ 71.4 \end{Bmatrix}$$

$$\det \mathbf{A} = 3\{7 \times 10 - (-0.3) \times (-0.2)\} + 0.1\{0.1 \times 10 - (-0.3) \times 0.3\} - 0.2\{0.1 \times (-0.2) - 7 \times 0.3\}$$

$$\det \mathbf{A} = 210.353$$

# Metode Cramer

$$\mathbf{A}_1 = \begin{bmatrix} 7.85 & -0.1 & -0.2 \\ -19.3 & 7 & -0.3 \\ 71.4 & -0.2 & 10 \end{bmatrix}$$

$$\mathbf{A}_2 = \begin{bmatrix} 3 & 7.85 & -0.2 \\ 0.1 & -19.3 & -0.3 \\ 0.3 & 71.4 & 10 \end{bmatrix}$$

$$\mathbf{A}_3 = \begin{bmatrix} 3 & -0.1 & 7.85 \\ 0.1 & 7 & -19.3 \\ 0.3 & -0.2 & 71.4 \end{bmatrix}$$

$$\det \mathbf{A}_1 = |\mathbf{A}_1| = 631.059$$

$$\det \mathbf{A}_2 = |\mathbf{A}_2| = -525.8825$$

$$\det \mathbf{A}_3 = |\mathbf{A}_3| = 1472.471$$

$$x_1 = \frac{\det \mathbf{A}_1}{\det \mathbf{A}} = \frac{631.058}{210.353} = 3$$

$$x_2 = \frac{\det \mathbf{A}_2}{\det \mathbf{A}} = \frac{-525.8825}{210.353} = -2.5$$

$$x_3 = \frac{\det \mathbf{A}_3}{\det \mathbf{A}} = \frac{1472.471}{210.353} = 7$$

# Metode Eliminasi

## □ Contoh: 2 persamaan linear

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 = c_1 &\Rightarrow a_{21}(a_{11}x_1 + a_{12}x_2 = c_1) \Rightarrow a_{21}a_{11}x_1 + a_{21}a_{12}x_2 = a_{21}c_1 \\ a_{21}x_1 + a_{22}x_2 = c_2 &\Rightarrow a_{11}(a_{21}x_1 + a_{22}x_2 = c_2) \Rightarrow a_{11}a_{21}x_1 + a_{11}a_{22}x_2 = a_{11}c_2 \end{aligned}$$

---

$$a_{21}a_{12}x_2 - a_{11}a_{22}x_2 = a_{21}c_1 - a_{11}c_2$$

$$x_2 = \frac{a_{21}c_1 - a_{11}c_2}{a_{21}a_{12} - a_{11}a_{22}}$$

$$x_1 = \frac{c_1 - a_{12}x_2}{a_{11}} = \frac{c_1}{a_{11}} - \frac{a_{12}}{a_{11}} \left( \frac{a_{21}c_1 - a_{11}c_2}{a_{21}a_{12} - a_{11}a_{22}} \right)$$

# Eliminasi Gauss

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## □ Strategi

- *Forward elimination*
- *Back substitution*

## □ Contoh

- 3 persamaan linear  $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = c_1$  (1)  
 $a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = c_2$  (2)  
 $a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = c_3$  (3)

# Eliminasi Gauss

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## □ Forward elimination #1

- Hilangkan  $x_1$  dari pers. kedua dan ketiga dengan operasi perkalian koefisien dan pengurangan terhadap pers. pertama.

*pivot coefficient* →  $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = c_1$  (pivot equation)

$$\left(a_{22} - \frac{a_{21}}{a_{11}}a_{12}\right)x_2 + \left(a_{23} - \frac{a_{21}}{a_{11}}a_{13}\right)x_3 = c_2 - \frac{a_{21}}{a_{11}}c_1$$
$$\left(a_{32} - \frac{a_{31}}{a_{11}}a_{12}\right)x_2 + \left(a_{33} - \frac{a_{31}}{a_{11}}a_{13}\right)x_3 = c_3 - \frac{a_{31}}{a_{11}}c_1$$

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = c_1$  (1)

$a'_{22}x_2 + a'_{23}x_3 = c'_2$  (2')

$a'_{32}x_2 + a'_{33}x_3 = c'_3$  (3')

# Eliminasi Gauss

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## □ Forward elimination #2

- Hilangkan  $x_2$  dari pers. ketiga dengan operasi perkalian koefisien dan pengurangan terhadap pers. kedua.

*pivot coefficient*

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = c_1$$

$$a'_{22}x_2 + a'_{23}x_3 = c'_2$$

$$\left( a'_{33} - \frac{a'_{32}}{a'_{22}} a'_{23} \right) x_3 = c'_3 - \frac{a'_{32}}{a'_{22}} c'_2$$

*pivot equation*



$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = c_1 \quad (1)$$

$$a'_{22}x_2 + a'_{23}x_3 = c'_2 \quad (2')$$

$$a''_{33}x_3 = c''_3 \quad (3'')$$

# Eliminasi Gauss

## □ *Back substitution*

- Hitung  $x_3$  dari pers. (3''), hitung  $x_2$  dari pers. (2'), dan  $x_1$  dari pers. (1)

$$x_3 = \frac{c_3''}{a_{33}''} \quad \Rightarrow \quad x_2 = \frac{c_2' - a_{23}'x_3}{a_{22}'} \quad \Rightarrow \quad x_1 = \frac{c_1 - a_{12}x_2 - a_{13}x_3}{a_{11}}$$

# Eliminasi Gauss

## □ Forward elimination

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = c_1$$

$$a'_{22}x_2 + a'_{23}x_3 + \cdots + a'_{2n}x_n = c'_2$$

$$a''_{33}x_3 + \cdots + a''_{3n}x_n = c''_3$$

.

.

.

$$a_{nn}^{n-1}x_n = c_n^{n-1}$$

## □ Back substitution

$$x_n = \frac{c_n^{n-1}}{a_{nn}^{n-1}}$$

$$x_i = \frac{c_i^{i-1} - \sum_{j=i+1} a_{ij}^{i-1} x_j}{a_{ii}^{i-1}}, \quad i = n-1, n-2, \dots, 1$$

# Eliminasi Gauss

- Contoh: 3 persamaan linear

$$(1) \quad 3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$

$$(2) \quad 0.1x_1 + 7x_2 - 0.3x_3 = -19.3$$

$$(3) \quad 0.3x_1 - 0.2x_2 + 10x_3 = 71.4$$

# Eliminasi Gauss

## □ *Forward elimination*

- Eliminasi  $x_1$  dari Pers. 2 dan 3, Pers. 1 sebagai *pivot*

$$(1) \quad 3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$

$$(2') \quad 0x_1 + 7.0033x_2 - 0.2933x_3 = -19.5617$$

$$(3') \quad 0x_1 - 0.19x_2 + 10.02x_3 = 70.615$$

- Eliminasi  $x_2$  dari Pers. 3, Pers. 2 sebagai *pivot*

$$(1) \quad 3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$

$$(2') \quad 0x_1 + 7.0033x_2 - 0.2933x_3 = -19.5617$$

$$(3'') \quad 0x_1 + 0x_2 + 10.0120x_3 = 70.0843$$

# Eliminasi Gauss

## □ *Back substitution*

- Menghitung  $x_3$  dari Pers. 3''

$$x_3 = \frac{70.0843}{10.0120} = 7$$

- Substitusi  $x_3$  ke Pers. 2' untuk menghitung  $x_2$

$$x_2 = \frac{-19.5617 + 0.2933 \times 7}{7.0033} = -2.5$$

- Substitusi  $x_3$  dan  $x_2$  ke Pers. 1 untuk menghitung  $x_1$

$$x_1 = \frac{7.85 + 0.2 \times 7 + 0.1 \times (-2.5)}{3} = 3$$

# Metode Eliminasi

- ❑ Strategi
  - ❑ Eliminasi variabel tak diketahui,  $x_j$ , dengan penggabungan dua persamaan.
  - ❑ Hasil eliminasi adalah satu persamaan yang dapat diselesaikan untuk mendapatkan satu variabel  $x_j$ .

# Kelemahan Metode Eliminasi

- ❑ Pembagian dengan nol
  - ❑ *Pivot coefficient* sama dengan nol ataupun sangat kecil.
  - ❑ Pembagian dengan nol dapat terjadi selama proses eliminasi ataupun substitusi.
- ❑ *Round-off errors*
  - ❑ Selama proses eliminasi maupun substitusi, setiap langkah hitungan bergantung pada langkah hitungan sebelumnya dan setiap kali terjadi kesalahan; kesalahan dapat berakumulasi, terutama apabila jumlah persamaan sangat banyak.
- ❑ *Ill-conditioned systems*
  - ❑ *Ill-condition* adalah situasi dimana perubahan kecil pada satu atau beberapa koefisien berakibat perubahan yang besar pada hasil hitungan.

# Perbaikan

- ❑ Pemilihan *pivot* (*pivoting*)
  - ❑ Urutan persamaan dipilih sedemikian hingga yang menjadi *pivot equation* adalah persamaan yang memberikan *pivot coefficient* terbesar.

# Metode Gauss-Jordan

- Mirip dengan metode eliminasi Gauss, tetapi tidak diperlukan *back substitution*.
- Contoh
  - 3 persamaan linear
    - (1)  $3x_1 - 0.1x_2 - 0.2x_3 = 7.85$
    - (2)  $0.1x_1 + 7x_2 - 0.3x_3 = -19.3$
    - (3)  $0.3x_1 - 0.2x_2 + 10x_3 = 71.4$

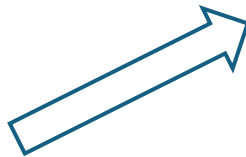
# Metode Gauss-Jordan

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$$\left[ \begin{array}{ccc|c} 3 & -0.1 & -0.2 & 7.85 \\ 0.1 & 7 & -0.3 & -19.3 \\ 0.3 & -0.1 & 10 & 71.4 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 1 & -0.0333 & -0.0667 & 2.6167 \\ 0.1 & 7 & -0.3 & -19.3 \\ 0.3 & -0.1 & 10 & 71.4 \end{array} \right]$$



$$\left[ \begin{array}{ccc|c} 3/3 & -0.1/3 & -0.2/3 & 7.85/3 \\ 0.1 & 7 & -0.3 & -19.3 \\ 0.3 & -0.1 & 10 & 71.4 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 1 & -0.0333 & -0.0667 & 2.6167 \\ 0 & 7.0033 & -0.2933 & -19.5617 \\ 0 & -0.1900 & 10.0200 & 70.6150 \end{array} \right]$$

# Metode Gauss-Jordan

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$$\left[ \begin{array}{ccc|c} 1 & -0.0333 & -0.0667 & 2.6167 \\ 0 & 7.0033 & -0.2933 & -19.5617 \\ 0 & -0.1900 & 10.0200 & 70.6150 \end{array} \right]$$



$$\left[ \begin{array}{ccc|c} 1 & -0.0333 & -0.0667 & 2.6167 \\ 0/7.0033 & 7.0033/7.0033 & -0.2933/7.0033 & -19.5617/7.0033 \\ 0 & -0.1900 & 10.0200 & 70.6150 \end{array} \right]$$



$$\left[ \begin{array}{ccc|c} 1 & -0.0333 & -0.0667 & 2.6167 \\ 0 & 1 & -0.0419 & -2.7931 \\ 0 & -0.1900 & 10.0200 & 70.6150 \end{array} \right]$$



$$\left[ \begin{array}{ccc|c} 1 & 0 & -0.0681 & 2.5236 \\ 0 & 1 & -0.0419 & -2.7931 \\ 0 & 0 & 10.0120 & 70.0843 \end{array} \right]$$

# Metode Gauss-Jordan

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$$\left[ \begin{array}{ccc|c} 1 & 0 & -0.0681 & 2.5236 \\ 0 & 1 & -0.0419 & -2.7931 \\ 0/10.0120 & 0/10.0120 & 10.0120/10.0120 & 70.0843/10.0120 \end{array} \right]$$



$$\begin{array}{l} \rightarrow \\ \rightarrow \\ \rightarrow \end{array} \left[ \begin{array}{ccc|c} 1 & 0 & -0.0681 & 2.5236 \\ 0 & 1 & -0.0419 & -2.7931 \\ 0 & 0 & 1 & 7 \end{array} \right] \Rightarrow \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2.5 \\ 0 & 0 & 1 & 7 \end{array} \right] \Rightarrow \begin{cases} x_1 \\ x_2 \\ x_3 \end{cases} = \begin{cases} 3 \\ -2.5 \\ 7 \end{cases}$$

# Gauss-Jordan vs Eliminasi Gauss

- ❑ Metode Gauss-Jordan
  - ❑ Jumlah operasi lebih banyak (50%)
  - ❑ Memiliki kelemahan yang sama dengan eliminasi Gauss
    - Pembagian dengan nol
    - *Round-off error*

# LU decomposition

- Gauss elimination  $\rightarrow$  LU decomposition

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{[A]} \rightarrow \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ f_{21} & 1 & 0 \\ f_{31} & f_{32} & 1 \end{bmatrix}}_{\{L\}} \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix}}_{[U]}$$

$$f_{21} = \frac{a_{21}}{a_{11}}$$
$$f_{31} = \frac{a_{31}}{a_{11}} \quad f_{32} = \frac{a'_{32}}{a'_{22}}$$

# LU decomposition

$$(1) \quad [L][U] = [A] \quad \longrightarrow$$

$$(2) \quad [L]\{D\} = \{C\} \quad \longrightarrow$$

$$(3) \quad [U]\{X\} = \{D\} \quad \longrightarrow$$

## Procedure

- ❑ Construct  $[U]$  —Step 1
- ❑ Construct  $[L]$  —Step 2
- ❑ Get  $\{D\}$  —Step 3
- ❑ Obtain  $\{X\}$  —Step 4

# LU decomposition

- Steps 1: construct  $[U]$  following Gauss elimination

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & a'_{32} & a'_{33} \end{bmatrix} \rightarrow \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix}}_{[U]}$$

- Step 2: construct  $[L]$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \rightarrow \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ f_{21} & 1 & 0 \\ f_{31} & f_{32} & 1 \end{bmatrix}}_{[L]} \quad \begin{aligned} f_{21} &= \frac{a_{21}}{a_{11}} \\ f_{31} &= \frac{a_{31}}{a_{11}} & f_{32} &= \frac{a'_{32}}{a'_{22}} \end{aligned}$$

# LU decomposition

- Step 3: get  $\{D\}$

$$[L]\{D\} = \{C\}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ f_{21} & 1 & 0 \\ f_{31} & f_{32} & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix}$$

$$d_1 = c_1$$

$$\begin{aligned} f_{21}d_1 + d_2 &= c_2 \\ \Leftrightarrow d_2 &= c_2 - f_{21}d_1 \end{aligned}$$

$$\begin{aligned} f_{31}d_1 + f_{32}d_2 + d_3 &= c_3 \\ \Leftrightarrow d_3 &= c_3 - f_{31}d_1 - f_{32}d_2 \end{aligned}$$

- Step 4: obtain  $\{X\}$

$$[U]\{X\} = \{D\}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix}$$

$$\begin{aligned} a''_{33}x_3 &= d_3 \\ \boxed{x_3} &\Leftrightarrow x_3 = d_3/a''_{33} \end{aligned}$$

$$\begin{aligned} a'_{22}x_2 + a'_{23}x_3 &= d_2 \\ \boxed{x_2} &\Leftrightarrow x_2 = (d_2 - a'_{23}x_3)/a'_{22} \end{aligned}$$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= d_1 \\ \boxed{x_1} &\Leftrightarrow x_1 = (d_1 - a_{12}x_2 - a_{13}x_3)/a_{11} \end{aligned}$$

# LU decomposition

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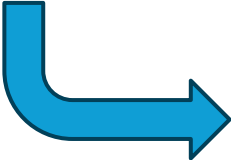
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## □ Example

$$(1) \quad 3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$

$$(2) \quad 0.1x_1 + 7x_2 - 0.3x_3 = -19.3$$

$$(3) \quad 0.3x_1 - 0.2x_2 + 10x_3 = 71.4$$


$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0.1 & 7 & -0.3 \\ 0.3 & -0.2 & 10 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.3 \\ 71.4 \end{Bmatrix}$$

$[A] \qquad \{X\} \qquad \{C\}$

# LU decomposition

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<https://istiarto.staff.ugm.ac.id>

$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0.1 & 7 & -0.3 \\ 0.3 & -0.2 & 10 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & -0.1 & -0.2 \\ 0 & 7.0033 & -0.2933 \\ 0 & -0.19 & 10.0200 \end{bmatrix} \Rightarrow [U] = \begin{bmatrix} 3 & -0.1 & -0.2 \\ 0 & 7.0033 & -0.2933 \\ 0 & 0 & 10.0120 \end{bmatrix}$$

$$f_{21} = \frac{a_{21}}{a_{11}} = \frac{0.1}{3} = 0.0333$$

$$f_{31} = \frac{a_{31}}{a_{11}} = \frac{0.3}{3} = 0.1$$

$$f_{32} = \frac{a'_{32}}{a'_{22}} = \frac{-0.19}{7.0033} = -0.0271$$



$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ 0.0333 & 1 & 0 \\ 0.1000 & -0.0271 & 1 \end{bmatrix}$$

# *LU decomposition*

$$[L]\{D\} = \{C\}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.0333 & 1 & 0 \\ 0.1000 & 0.0271 & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.3 \\ 71.4 \end{Bmatrix}$$

$$d_1 = 7.85$$

$$0.0333d_1 + d_2 = -19.3 \Leftrightarrow d_2 = -19.3 - 0.0333(7.85) = -19.5617$$

$$0.1d_1 + 0.0271d_2 + d_3 = 71.4$$

$$\Leftrightarrow d_3 = 71.4 - 0.1(7.85) - 0.0271(-19.5617) = 70.0843$$

# LU decomposition

$$[U]\{X\} = \{D\}$$


$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0 & 7.0033 & -0.2933 \\ 0 & 0 & 10.0120 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.5617 \\ 70.0843 \end{Bmatrix}$$

$$10.0120x_3 = 70.0843 \Leftrightarrow x_3 = \frac{70.0843}{10.0120} = 7$$

$$7.0033x_2 - 0.2933x_3 = -19.5617 \Leftrightarrow x_2 = \frac{-19.5617 + 0.2933(7)}{7.0033} = -2.5$$

$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$

$$\Leftrightarrow x_1 = \frac{7.85 + 0.1(-2.5) + 0.2(7)}{3} = 3$$


$$\{X\} = \begin{Bmatrix} 3 \\ -2.5 \\ 7 \end{Bmatrix}$$

# Matriks Inversi

$$[A] \cdot \{X\} = \{C\} \Rightarrow \{X\} = [A]^{-1} \cdot \{C\}$$

$$\left[ \begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array} \right] \Rightarrow \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & a_{11}^{-1} & a_{12}^{-1} & a_{13}^{-1} \\ 0 & 1 & 0 & a_{21}^{-1} & a_{22}^{-1} & a_{23}^{-1} \\ 0 & 0 & 1 & a_{31}^{-1} & a_{32}^{-1} & a_{33}^{-1} \end{array} \right]$$

# Matriks Inversi

□ Contoh: 3 persamaan linear

$$(1) \quad 3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$

$$(2) \quad 0.1x_1 + 7x_2 - 0.3x_3 = -19.3$$

$$(3) \quad 0.3x_1 - 0.2x_2 + 10x_3 = 71.4$$

# Matriks Inversi

$$[A] = \left[ \begin{array}{ccc|ccc} 3 & -0.1 & -0.2 & 1 & 0 & 0 \\ 0.1 & 7 & -0.3 & 0 & 1 & 0 \\ 0.3 & -0.2 & 10 & 0 & 0 & 1 \end{array} \right] \Rightarrow [A] = \left[ \begin{array}{ccc|ccc} 1 & -0.0333 & -0.0667 & 0.3333 & 0 & 0 \\ 0.1 & 7 & -0.3 & 0 & 1 & 0 \\ 0.3 & -0.2 & 10 & 0 & 0 & 1 \end{array} \right]$$



$$[A] = \left[ \begin{array}{ccc|ccc} 1 & -0.0333 & -0.0667 & 0.3333 & 0 & 0 \\ 0 & 7.0033 & -0.2933 & -0.0333 & 1 & 0 \\ 0 & -0.1900 & 10.0200 & -0.0999 & 0 & 1 \end{array} \right]$$

# Matriks Inversi

$$[A] = \left[ \begin{array}{ccc|ccc} 1 & -0.0333 & -0.0667 & 0.3333 & 0 & 0 \\ 0 & 1 & -0.0417 & -0.0047 & 0.1422 & 0 \\ 0 & -0.1900 & 10.0200 & -0.0999 & 0 & 1 \end{array} \right]$$




$$[A] = \left[ \begin{array}{ccc|ccc} 1 & 0 & -0.0681 & 0.3318 & 0.0047 & 0 \\ 0 & 1 & -0.0417 & -0.0047 & 0.1422 & 0 \\ 0 & 0 & 10.0121 & -0.1009 & 0.0270 & 1 \end{array} \right]$$

# Matriks Inversi

$$[A] = \left[ \begin{array}{ccc|ccc} 1 & 0 & -0.0681 & 0.3318 & 0.0047 & 0 \\ 0 & 1 & -0.0417 & -0.0047 & 0.1422 & 0 \\ 0 & 0 & 1 & -0.0101 & 0.0027 & 0.0999 \end{array} \right]$$



$$[A] = \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 0.3325 & 0.0049 & 0.0068 \\ 0 & 1 & 0 & -0.0052 & 0.1423 & 0.0042 \\ 0 & 0 & 1 & -0.0101 & 0.0027 & 0.0999 \end{array} \right]$$


$$[A]^{-1}$$

# Matriks Inversi

$$\{X\} = [A]^{-1} \cdot \{C\}$$

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} 0.3325 & 0.0049 & 0.0068 \\ -0.0052 & 0.1423 & 0.0042 \\ -0.0101 & 0.0027 & 0.0999 \end{bmatrix} \cdot \begin{Bmatrix} 7.85 \\ -19.3 \\ 71.4 \end{Bmatrix}$$

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 3.0004 \\ -2.4881 \\ 7.0002 \end{Bmatrix}$$

# Matriks Inversi

- ❑ Fungsi MS Excel untuk menghitung inversi sebuah matriks
  - ❑ `=MINVERSE()`
- ❑ Fungsi MS Excel untuk menghitung perkalian dua buah matriks
  - ❑ `=MMULT()`
  
- ❑ Ada kemungkinan matriks tidak memiliki inversi

# Metode Iteratif: Jacobi

$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= c_1 \\a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= c_2 \\a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= c_3\end{aligned}$$

$$\begin{aligned}x_1^0 &= 0 && \text{nilai awal,} \\x_2^0 &= 0 && \text{biasanya } x_i^0 = 0 \\x_3^0 &= 0\end{aligned}$$

iterasi diteruskan  
sampai konvergen

$$x_i^{n+1} \approx x_i^n, \forall x_i$$

$$\begin{aligned}x_1 &= \frac{c_1 - a_{12}x_2 - a_{13}x_3}{a_{11}} \\x_2 &= \frac{c_2 - a_{21}x_1 - a_{23}x_3}{a_{22}} \\x_3 &= \frac{c_3 - a_{31}x_1 - a_{32}x_2}{a_{33}}\end{aligned}$$

$$\begin{aligned}x_1^1 &= \frac{c_1 - a_{12}x_2^0 - a_{13}x_3^0}{a_{11}} \\x_2^1 &= \frac{c_2 - a_{21}x_1^0 - a_{23}x_3^0}{a_{22}} \\x_3^1 &= \frac{c_3 - a_{31}x_1^0 - a_{32}x_2^0}{a_{33}}\end{aligned}$$

$$\begin{aligned}x_1^{n+1} &= \frac{c_1 - a_{12}x_2^n - a_{13}x_3^n}{a_{11}} \\x_2^{n+1} &= \frac{c_2 - a_{21}x_1^n - a_{23}x_3^n}{a_{22}} \\x_3^{n+1} &= \frac{c_3 - a_{31}x_1^n - a_{32}x_2^n}{a_{33}}\end{aligned}$$

# Metode Iteratif: Jacobi

- Contoh: 3 persamaan linear

$$(1) \quad 3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$

$$(2) \quad 0.1x_1 + 7x_2 - 0.3x_3 = -19.3$$

$$(3) \quad 0.3x_1 - 0.2x_2 + 10x_3 = 71.4$$

# Metode Iteratif: Gauss-Seidel

$$x_1^1 = \frac{c_1 - a_{12}x_2^0 - a_{13}x_3^0}{a_{11}}$$
$$x_2^1 = \frac{c_2 - a_{21}x_1^0 - a_{23}x_3^0}{a_{22}}$$
$$x_3^1 = \frac{c_1 - a_{31}x_1^0 - a_{32}x_2^0}{a_{33}}$$



$$x_1^{n+1} = \frac{c_1 - a_{12}x_2^n - a_{13}x_3^n}{a_{11}}$$
$$x_2^{n+1} = \frac{c_2 - a_{21}x_1^{n+1} - a_{23}x_3^n}{a_{22}}$$
$$x_3^{n+1} = \frac{c_1 - a_{31}x_1^{n+1} - a_{32}x_2^{n+1}}{a_{33}}$$

iterasi diteruskan  
sampai konvergen

$$x_i^{n+1} \approx x_i^n, \forall x_i$$

# Successive Over-relaxation Method

- Dalam setiap iterasi, nilai variabel terbaru (yang baru saja dihitung),  $x_{n+1}$ , tidak langsung dipakai dalam iterasi selanjutnya
- Dalam iterasi selanjutnya, nilai tsb dimodifikasi dengan memasukkan pengaruh nilai variabel lama (dalam iterasi sebelumnya),  $x_n$

$$x_i^{new} = \lambda x_i^{n+1} + (1 - \lambda)x_i^n$$

- faktor relaksasi  $\lambda$  dimaksudkan untuk mempercepat konvergensi hitungan (iterasi)
- *under-relaxation*     $0 < \lambda < 1$
- *over-relaxation*      $1 < \lambda < 2$

# Successive Over-relaxation Method

$$x_1^{n+1} = \frac{c_1 - a_{12}x_2^n - a_{13}x_3^n}{a_{11}}$$

$$x_2^{n+1} = \frac{c_2 - a_{21}[\lambda x_1^{n+1} + (1 - \lambda)x_1^n] - a_{23}x_3^n}{a_{22}}$$

$$x_3^{n+1} = \frac{c_3 - a_{31}[\lambda x_1^{n+1} + (1 - \lambda)x_1^n] - a_{32}[\lambda x_2^{n+1} + (1 - \lambda)x_2^n]}{a_{33}}$$

# Successive Over-relaxation Method

□ Contoh: 3 persamaan linear

$$(1) \quad 3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$

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Terima kasih