



Universitas Gadjah Mada
Departemen Teknik Sipil dan Lingkungan
Prodi Teknik Sipil (Program Sarjana)

INTEGRASI NUMERIS

Numerical Differentiation and Integration

Integrasi Numeris

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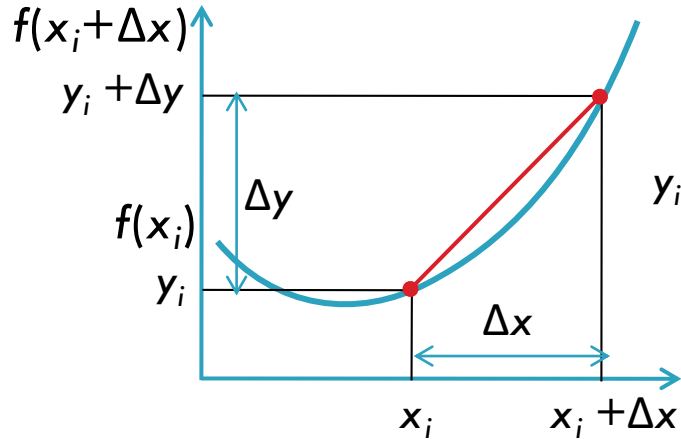
❑ Acuan

- ❑ Chapra, S.C., Canale R.P., 1990, *Numerical Methods for Engineers*, 2nd Ed., McGraw-Hill Book Co., New York.
 - Chapter 15 dan 16, hlm. 459-523.

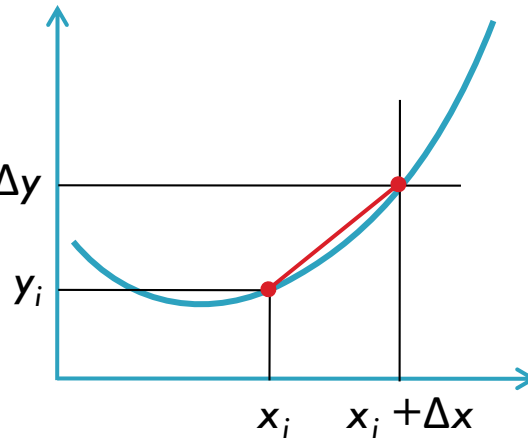
Diferensial, Derivatif

3

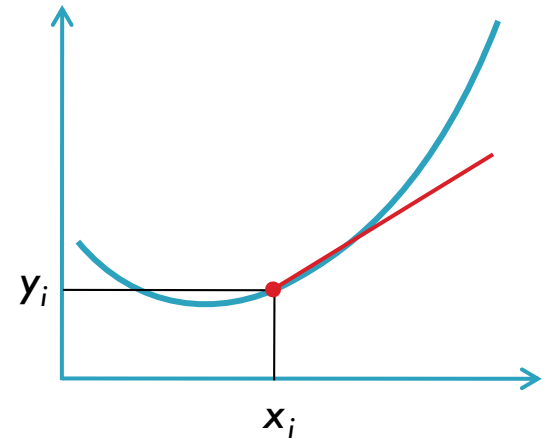
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(a)



(b)



(c)

$$\frac{\Delta y}{\Delta x} = \frac{f(x_i + \Delta x) - f(x_i)}{\Delta x}$$

difference approximation

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{f(x_i + \Delta x) - f(x_i)}{\Delta x}$$

Diferensial, Derivatif

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$$\frac{\Delta y}{\Delta x} = \frac{f(x_i + \Delta x) - f(x_i)}{\Delta x}$$



$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{f(x_i + \Delta x) - f(x_i)}{\Delta x}$$

pendekatan beda (hingga)
difference approximation

derivatif

Deret Taylor

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n$$

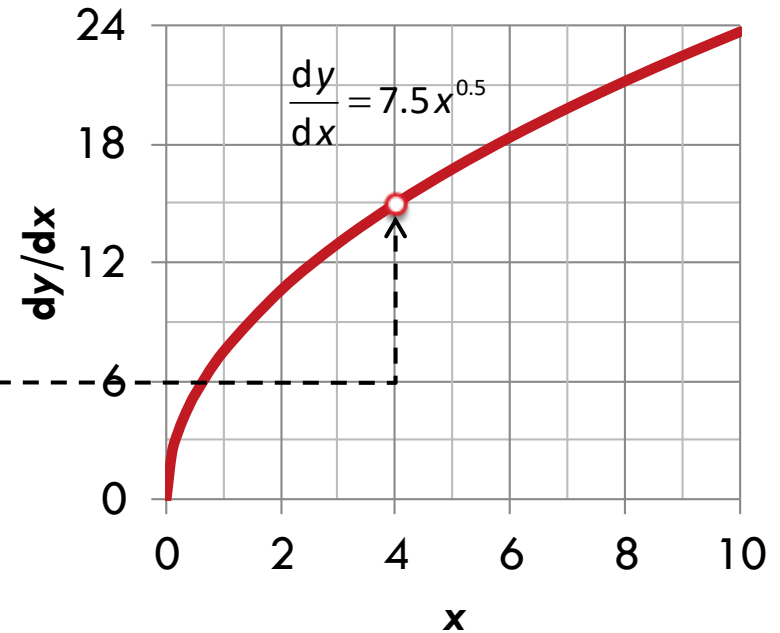
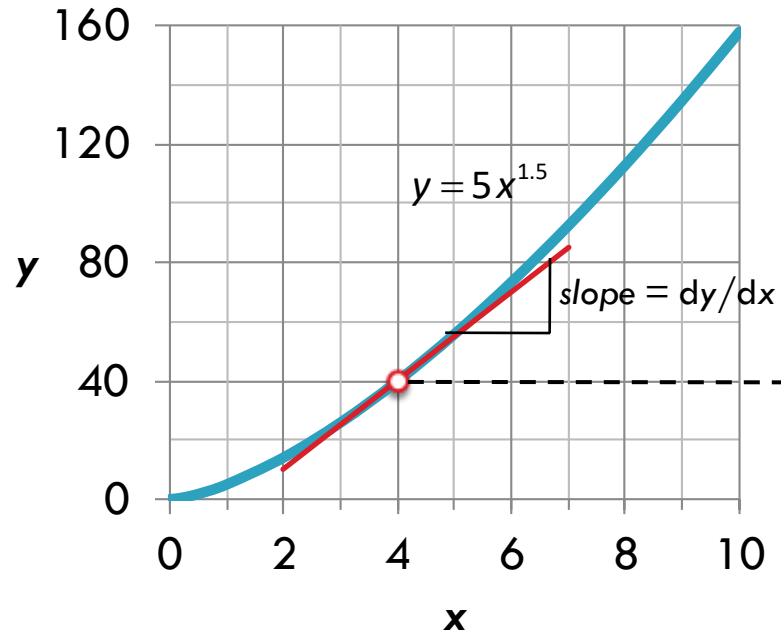
$$\frac{dy}{dx} = y' = f'(x)$$

derivatif = laju perubahan y
terhadap x

Diferensial

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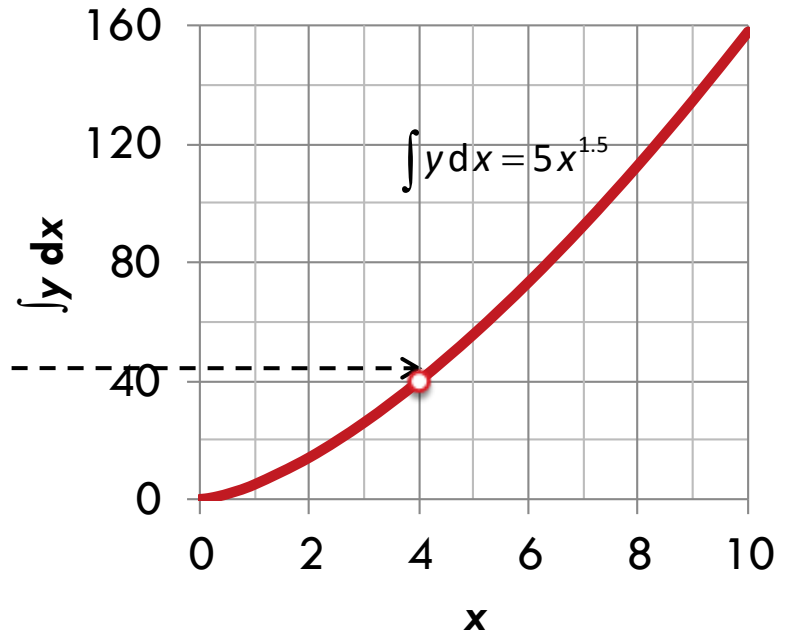
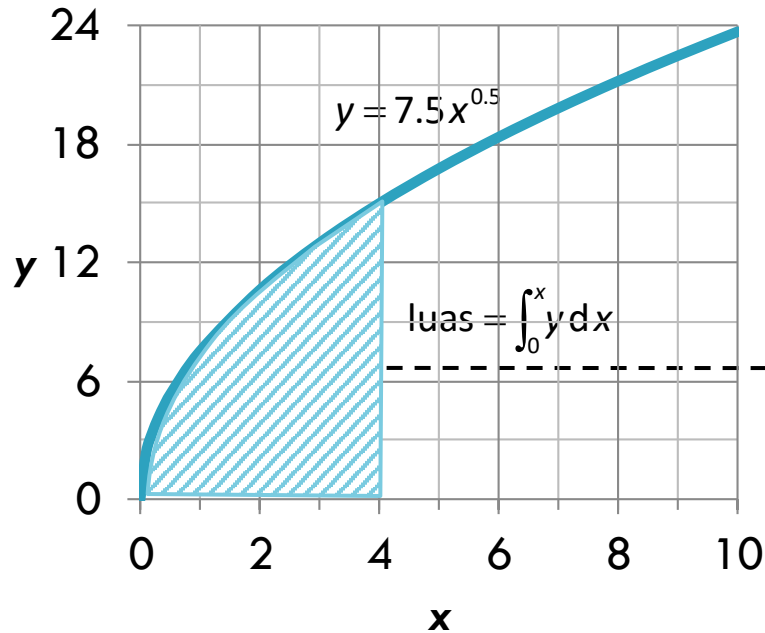
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Integral

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- “kebalikan” dari proses men-diferensial-kan adalah meng-integral-kan
- integrasi \times diferensiasi

Fungsi

- ❑ Fungsi-fungsi yang di-diferensial-kan atau di-integral-kan dapat berupa:
 - ❑ fungsi kontinu sederhana: polinomial, eksponensial, trigonometri
 - ❑ fungsi kontinu kompleks yang tidak memungkinkan didiferensialkan atau diintegrasikan secara langsung
 - ❑ fungsi yang nilai-nilainya disajikan dalam bentuk tabel [tabulasi data x vs $f(x)$]

Cara mencari nilai integral

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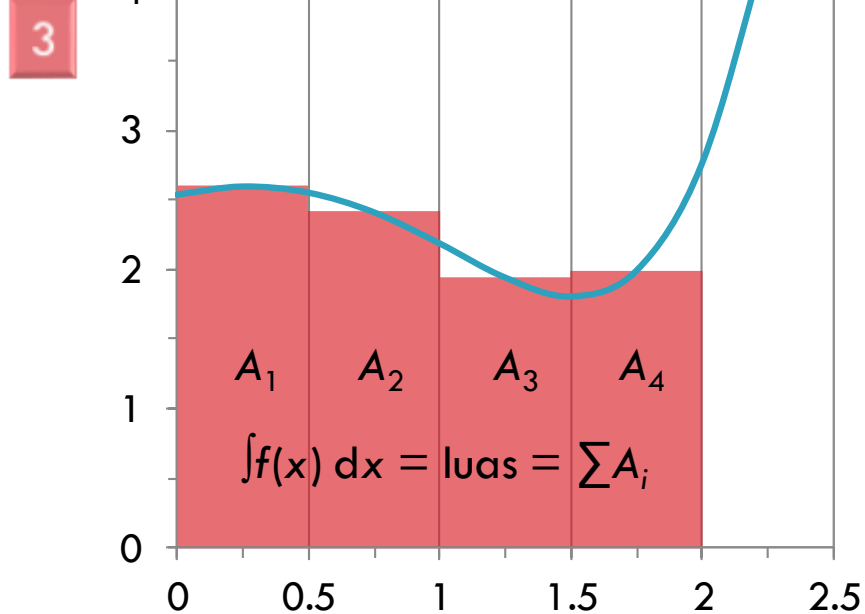
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1

$$\int_0^2 \frac{2 + \cos(1 + x^{3/2})}{\sqrt{1 + 0.5\sin x}} e^{0.5x} dx$$

2

x	f(x)
0.25	2.599
0.75	2.414
1.25	1.945
1.75	1.993



Derivatif

$$u = f(x) \text{ dan } v = f(x)$$

$$y = u^n \implies \frac{dy}{dx} = nu^{n-1} \frac{du}{dx}$$

$$y = uv \implies \frac{dy}{dx} = v \frac{du}{dx} + u \frac{dv}{dx}$$

$$y = \frac{u}{v} \implies \frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

$$\frac{d}{dx} \sin x = \cos x$$

$$\frac{d}{dx} \cos x = -\sin x$$

$$\frac{d}{dx} \tan x = \sec^2 x$$

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

$$\frac{d}{dx} e^x = e^x$$

$$\frac{d}{dx} \cot x = -\operatorname{csc}^2 x$$

$$\frac{d}{dx} \sec x = \sec x \tan x$$

$$\frac{d}{dx} \csc x = -\csc x \cot x$$

$$\frac{d}{dx} \log_a x = \frac{1}{x \ln a}$$

$$\frac{d}{dx} a^x = a^x \ln a$$

Integral

$$\int u dv = uv - \int v du$$

$$\int u^n dv = \frac{u^{n+1}}{n+1} + C \quad n \neq -1$$

$$\int a^{bx} dx = \frac{a^{bx}}{b \ln a} + C \quad a > 0, a \neq 1$$

$$\int \frac{dx}{x} = \ln|x| + C$$

$$\int \sin(ax + b) dx = -\frac{1}{a} \cos(ax + b) + C$$

$$\int \cos(ax + b) dx = \frac{1}{a} \sin(ax + b) + C$$

$$\int \ln|x| dx = x \ln|x| - x + C$$

$$\int e^{ax} dx = \frac{e^{ax}}{a} + C$$

$$\int x e^{ax} dx = \frac{e^{ax}}{a^2} (ax - 1) + C$$

$$\int \frac{dx}{a + bx^2} = \frac{1}{\sqrt{ab}} \tan^{-1} \frac{\sqrt{ab}}{a} x + C$$

Metode Integrasi Newton-Cotes

Metode Trapesium

Metode Simpson

Metode Kuadratur Gauss

Persamaan Newton-Cotes

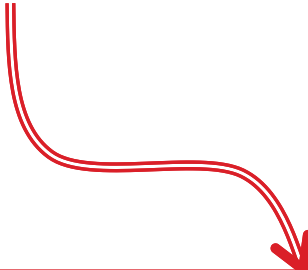
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□ Strategi

- mengganti fungsi kompleks dan rumit atau tabulasi data dengan **fungsi pendekatan** yang mudah untuk diintegrasikan

$$I = \int_a^b f(x) dx = \int_a^b f_n(x) dx$$

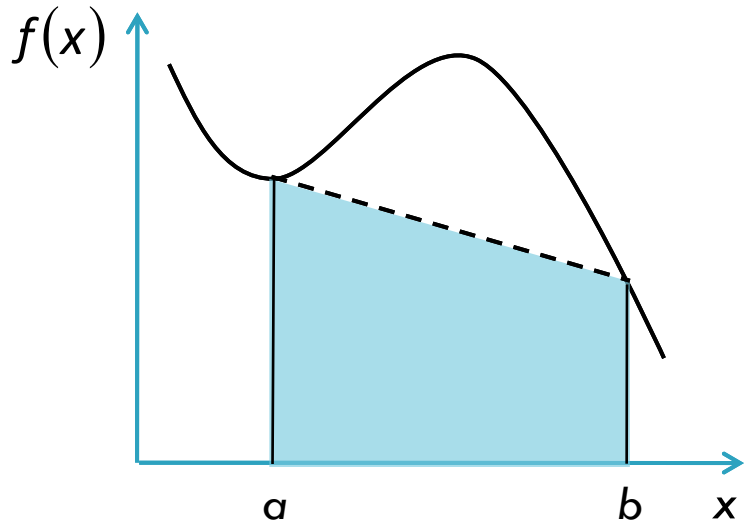

$$f_n(x) = a_0 + a_1x + a_2x^2 + \dots + a_{n-1}x^{n-1} + a_nx^n$$

polinomial tingkat n

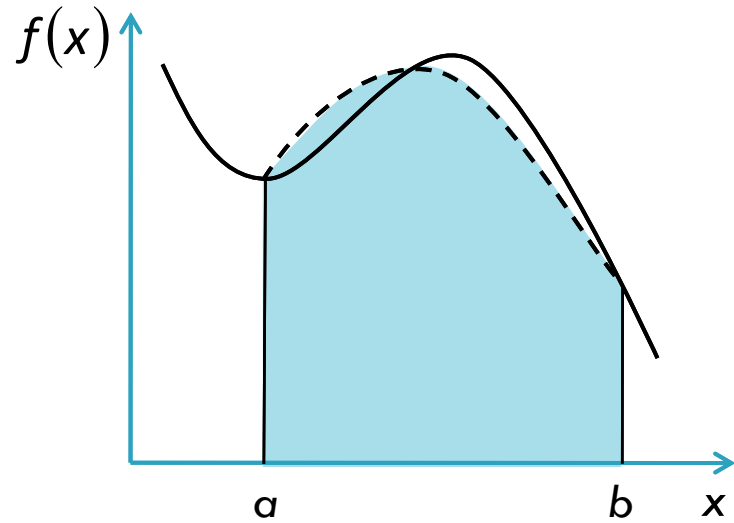
Persamaan Newton-Cotes

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Garis lurus (polinomial tingkat 1)
sbg fungsi pendekatan.

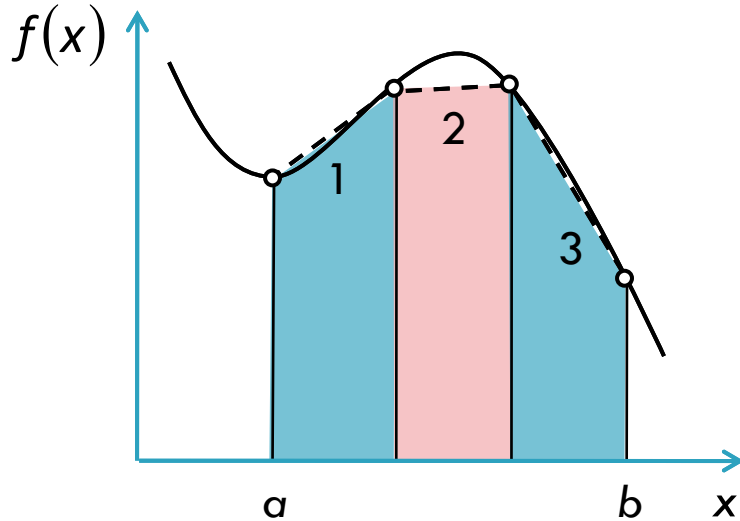


Kurva parabola (polinomial tingkat 2)
sbg fungsi pendekatan.

Persamaan Newton-Cotes

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Fungsi yang diintegrasikan didekati dengan 3 buah garis lurus (polinomial tingkat 1).

Dapat pula dipakai beberapa kurva polinomial tingkat yang lebih tinggi.

Garis lurus (polinomial tingkat 1) sbg fungsi pendekatan.

Metode Trapesium

- ❑ Fungsi pendekatan untuk menghitung integral adalah polinomial tingkat 1

$$I = \int_a^b f(x) dx = \int_a^b f_1(x) dx$$

- ❑ Sebuah garis lurus dapat dinyatakan dengan persamaan

$$f_1(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$

Metode Trapesium

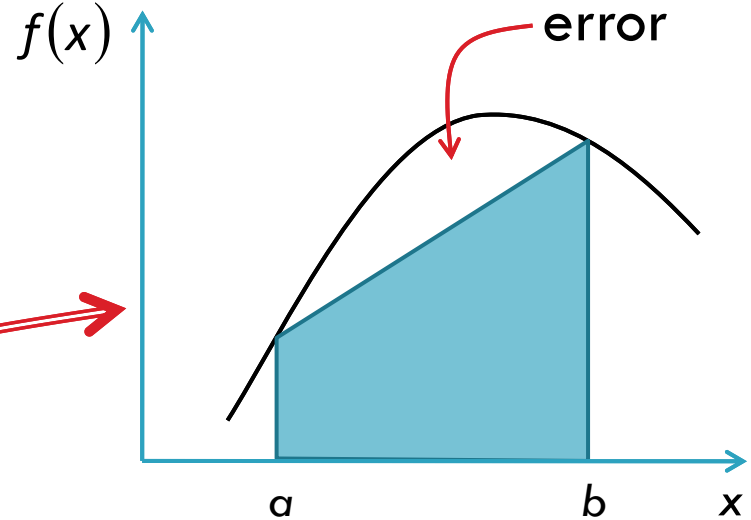
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$$I \cong \int_a^b \left[f(a) + \frac{f(b) - f(a)}{b - a} (x - a) \right] dx$$
$$\cong (b - a) \frac{f(a) + f(b)}{2}$$



Metode Trapesium



Metode Trapesium

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Penyelesaian eksak

$$I = \int_0^{0.8} (0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5) dx$$
$$= \left(0.2x + 12.5x^2 - \frac{200}{3}x^3 + \frac{675}{4}x^4 - 180x^5 + \frac{400}{6}x^6 \right) \Big|_0^{0.8} = 1.640533$$

Metode Trapesium

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

$$f(0) = 0.2 \text{ dan } f(0.8) = 0.232$$

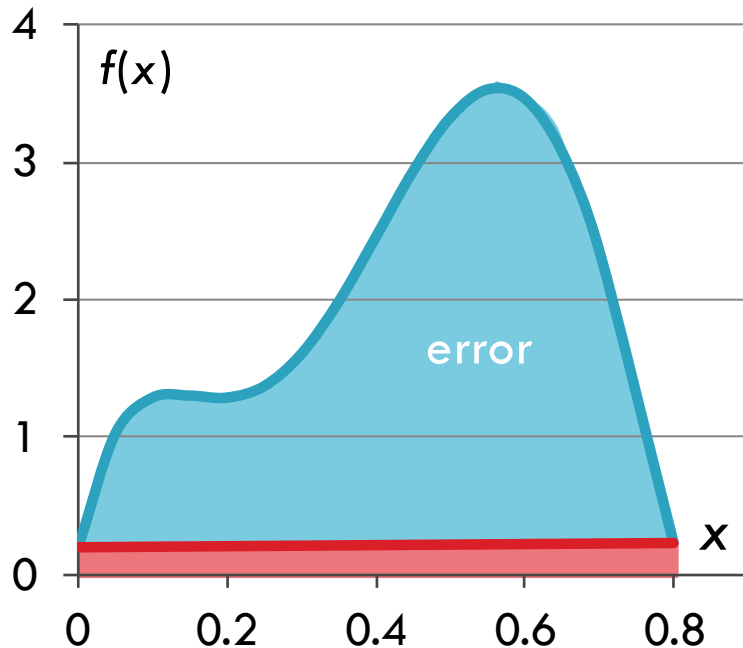
$$I = (0.8 - 0) \frac{0.2 + 0.232}{2} = 0.1728 \quad \Rightarrow \quad E_t = 1.640533 - 0.1728 = 1.467733 \text{ } (\approx 89\%) \text{ [error]}$$



Metode Trapesium

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- ❑ Error atau kesalahan
 - ❑ bentuk trapesium untuk menghitung nilai integral mengabaikan sejumlah besar porsi daerah di bawah kurva
- ❑ Kuantifikasi error pada Metode Trapesium

$$E_t = -\frac{1}{12} f''(\xi)(b-a)^3$$

ξ adalah titik di antara a dan b

Metode Trapesium

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$$f''(x) = -400 + 4050x^3 - 10800x^2 + 8000x^3$$

nilai rata-rata derivatif kedua:

$$\bar{f}''(x) = \frac{\int_0^{0.8} (-400 + 4050x^3 - 10800x^2 + 8000x^3) dx}{0.8 - 0} = -60$$

error:

$$E_a = -\frac{1}{12}(-0.60)(0.8 - 0)^3 = 2.56$$

order of magnitude nilai error ini sama dengan *order of magnitude* nilai error terhadap nilai penyelesaian eksak dan keduanya sama tanda (sama-sama positif)

Trapeسيوم multi pias

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❑ Peningkatan akurasi

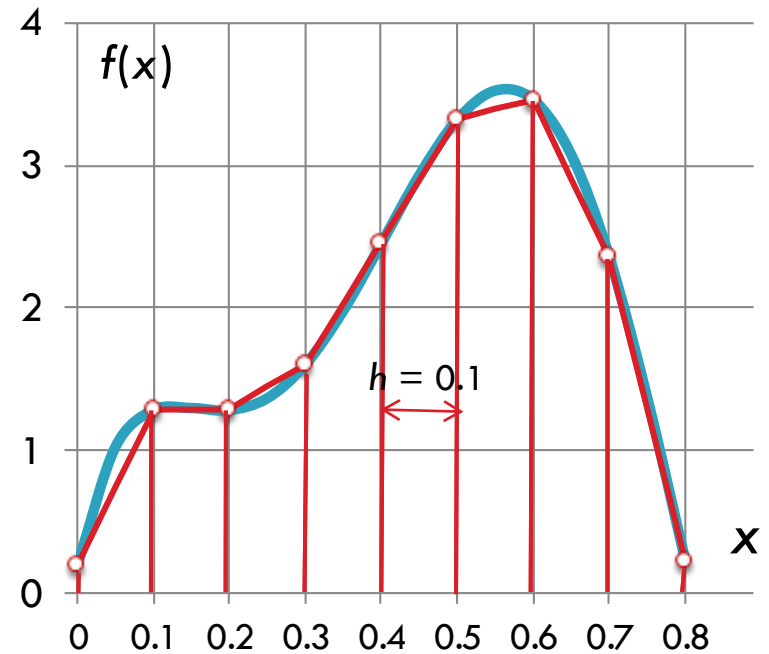
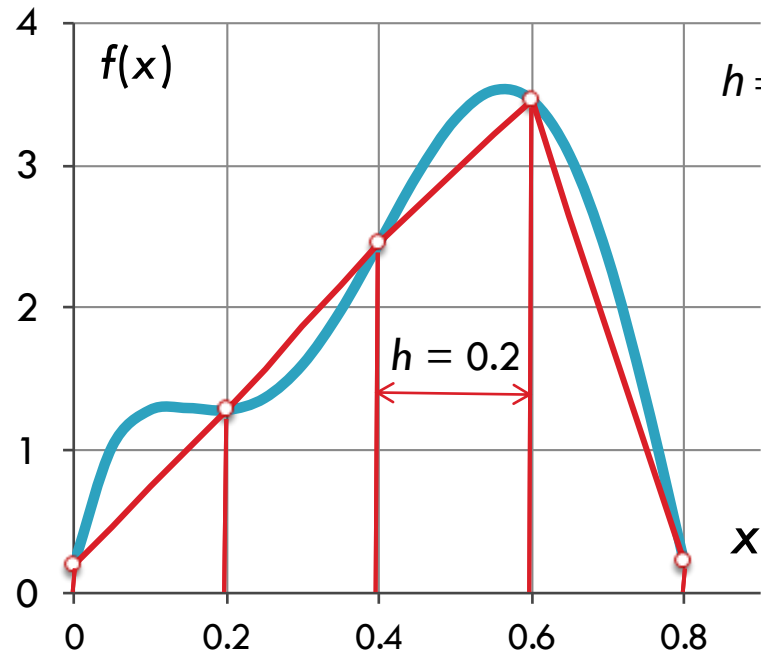
- ❑ selang ab dibagi menjadi sejumlah n pias dengan lebar seragam h

$$h = \frac{b-a}{n}$$

Trapesium multi pias

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Trapesium multi pias

$$h = \frac{b-a}{n} \quad \rightarrow \quad \text{Jika } a = x_0 \text{ dan } b = x_n$$

$$I = \int_{x_0}^{x_1} f(x) dx + \int_{x_1}^{x_2} f(x) dx + \dots + \int_{x_{n-1}}^{x_n} f(x) dx$$

$$I \approx h \frac{f(x_0) + f(x_1)}{2} + h \frac{f(x_1) + f(x_2)}{2} + \dots + h \frac{f(x_{n-1}) + f(x_n)}{2}$$

$$I \approx \frac{h}{2} \left[f(x_0) + \left\{ 2 \sum_{i=1}^{n-1} f(x_i) \right\} + f(x_n) \right] \quad \rightarrow \quad I \approx \underbrace{(b-a)}_{\text{lebar}} \underbrace{\frac{f(x_0) + \left\{ 2 \sum_{i=1}^{n-1} f(x_i) \right\} + f(x_n)}{2n}}_{\text{tinggi rata-rata}}$$

Trapeسيوم multi pias

Error = jumlah *error* pada setiap pias

$$E_t = -\frac{(b-a)^3}{12n^3} \sum_{i=1}^n f''(\xi_i)$$

$$\frac{1}{n} \sum_{i=1}^n f''(\xi_i) = \bar{f}'' \quad \Rightarrow \quad E_t \approx E_a = -\frac{(b-a)^3}{12n^2} \bar{f}''$$

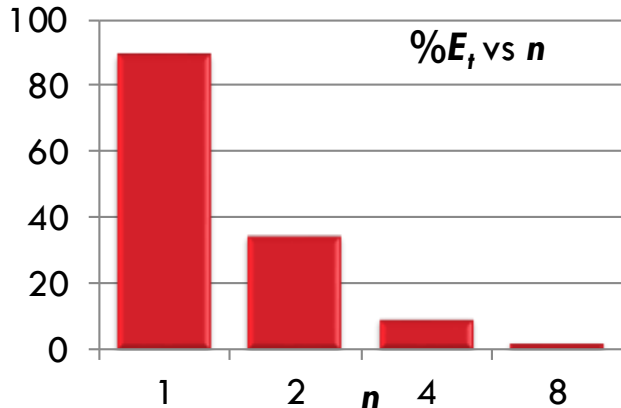
setiap kelipatan jumlah pias, *error* mengecil dengan faktor kuadrat peningkatan jumlah pias

Trapezium multi pias

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

$$I = \int_0^{0.8} f(x) dx = 1.640533 \quad (\text{exact solution})$$

$$I \approx \int_0^{0.8} f_1(x) dx \quad (\text{the trapezoidal rule})$$



<i>n</i>	<i>h</i>	<i>I</i>	<i>E_t</i>	% <i>E_t</i>	<i>E_a</i>
1	0.8	0.1728	1.4677	89%	2.56
2	0.4	1.0688	0.5717	35%	0.64
4	0.2	1.4848	0.1557	9%	0.16
8	0.1	1.6008	0.0397	2%	0.04

Metode Simpson

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The trapezoidal rule

- ❑ Fungsi pendekatan: polinomial tingkat 1
 - ❑ Peningkatan ketelitian dpt dilakukan dengan meningkatkan jumlah pias

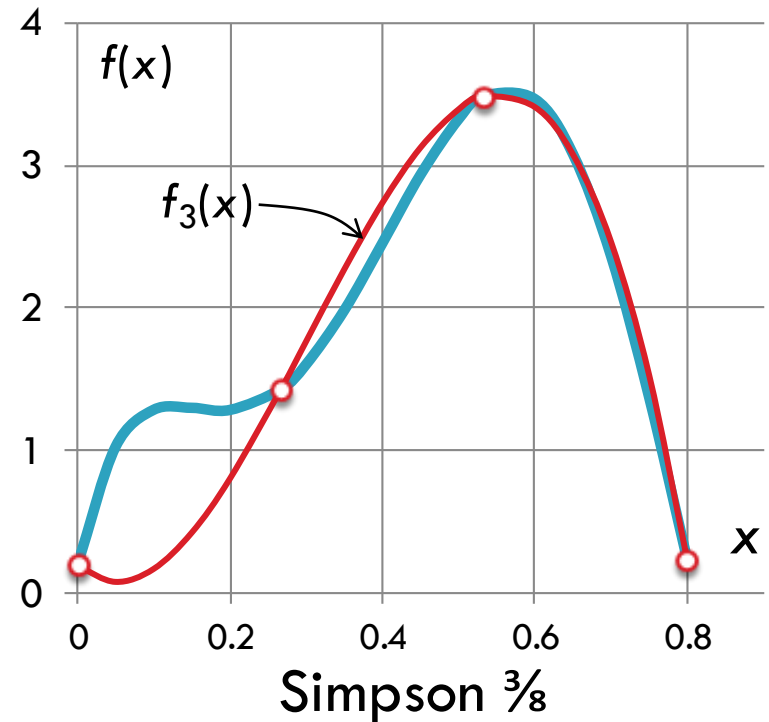
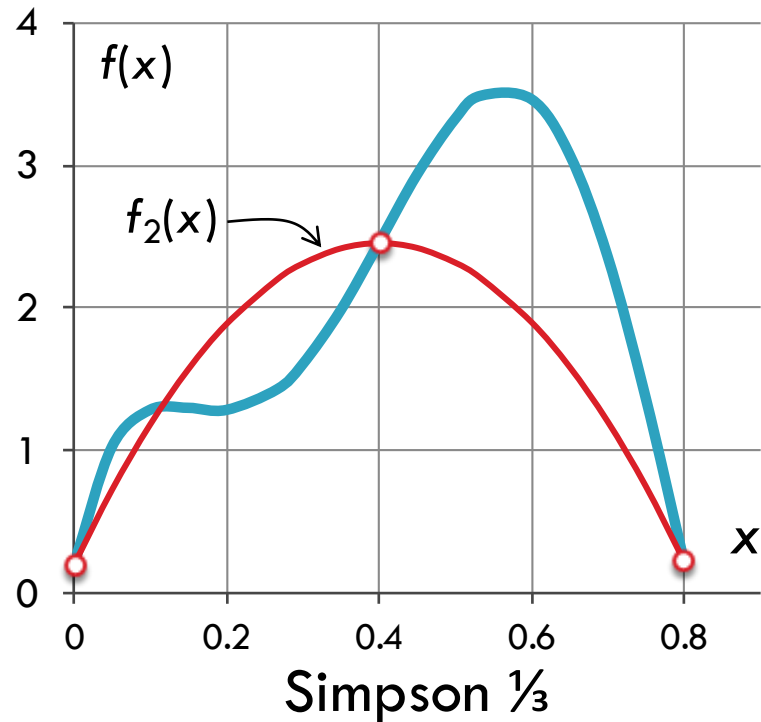
Simpson's rules

- ❑ Fungsi pendekatan: polinomial:
 - ❑ tingkat 2: Simpson 1/3
 - ❑ tingkat 3: Simpson 3/8

Metode Simpson

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Metode Simpson

- ❑ Polinomial tingkat 2 atau 3
 - ❑ dicari dengan Metode Newton atau Lagrange (lihat materi tentang *curve fitting*)

Simpson $\frac{1}{3}$

$$I = \int_a^b f(x) dx \approx \int_a^b f_2(x) dx$$

Jika $a = x_0$ dan $b = x_n$ dan $f_2(x)$ diperoleh dengan Metode Lagrange

$$I \approx \int_a^b \left[\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2) \right] dx$$

$$I \approx \frac{h}{3} [f(x_0) + 4f(x_1) + f(x_2)] \quad \text{atau} \quad I \approx (b-a) \frac{f(x_0) + 4f(x_1) + f(x_2)}{6}$$

$$h = \frac{b-a}{2}$$

$$E_t = -\frac{(b-a)^5}{180n^4} f^4(\xi)$$

Simpson $\frac{1}{3}$ multi pias

$$I \approx \int_{x_0}^{x_2} f(x) dx + \int_{x_2}^{x_4} f(x) dx + \dots + \int_{x_{n-2}}^{x_n} f(x) dx$$

$$I \approx 2h \frac{f(x_0) + 4f(x_1) + f(x_2)}{6} + 2h \frac{f(x_2) + 4f(x_3) + f(x_4)}{6} + \dots + 2h \frac{f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)}{6}$$

$$I \approx (b-a) \frac{f(x_0) + 4 \sum_{i=1,3,5}^{n-1} f(x_i) + 2 \sum_{i=2,4,6}^{n-2} f(x_i) + f(x_n)}{3n}$$

$$E_a = -\frac{(b-a)^5}{180n^4} \bar{f}^4 \quad (\text{estimasi error, } \bar{f}^4 \text{ rata-rata derivatif ke-4})$$

Simpson $\frac{3}{8}$

$$I = \int_a^b f(x) dx \approx \int_a^b f_3(x) dx$$

Jika $a = x_0$ dan $b = x_n$ dan $f_3(x)$ diperoleh dengan Metode Lagrange

$$I \approx \int_a^b \left[\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1) + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3) \right] dx$$

$$I \approx \frac{3h}{8} [f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)] \quad \text{atau} \quad I \approx \underbrace{(b-a)}_{\text{lebar}} \underbrace{\frac{f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)}{8}}_{\text{tinggi rata-rata}}$$

$$h = \frac{b-a}{3}$$

Simpson $\frac{3}{8}$

Error

$$E_t = -\frac{3}{80} h^5 f^4(\xi) \quad \text{atau} \quad E_t = -\frac{(b-a)^5}{6480} f^4(\xi)$$

Simpson $\frac{1}{3}$ dan $\frac{3}{8}$

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$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

$$I = \int_0^{0.8} f(x) dx = 1.640533 \quad (\text{exact solution})$$

Metode	I	E_r
Simpson $\frac{1}{3}$ ($n = 2$)	1.367467	0.273067 (27%)
Simpson $\frac{3}{8}$ ($n = 3$)	1.51917	0.121363 (7%)
Simpson $\frac{1}{3}$ ($n = 4$)	1.623467	0.017067 (1%)

Pias tak seragam: Metode Trapezium

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i	x_i	$f(x_i)$	I
0	0	0.2	
1	0.12	1.309729	0.090584
2	0.22	1.305241	0.130749
3	0.32	1.743393	0.152432
4	0.36	2.074903	0.076366
5	0.4	2.456	0.090618
6	0.44	2.842985	0.10598
7	0.54	3.507297	0.317514
8	0.64	3.181929	0.334461
9	0.7	2.363	0.166348
10	0.8	0.232	0.12975
			1.594801

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

I dihitung dengan Metode Trapezium di setiap pias:

$$I = h_1 \frac{f(x_1) + f(x_0)}{2} + h_2 \frac{f(x_2) + f(x_1)}{2} + \dots + h_n \frac{f(x_n) + f(x_{n-1})}{2}$$

$$\int_0^{0.8} f(x) dx = 1.594801$$

Pias tak seragam: Metode Simpson

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i	x_i	$f(x_i)$	I
0	0	0.2	
1	0.12	1.309729	
2	0.22	1.305241	
3	0.32	1.743393	
4	0.36	2.074903	
5	0.4	2.456	
6	0.44	2.842985	
7	0.54	3.507297	
8	0.64	3.181929	
9	0.7	2.363	
10	0.8	0.232	

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

I dihitung dengan Metode Simpson $\frac{1}{3}$ dan Simpson $\frac{3}{8}$:



PR, dikumpulkan minggu depan

Metode Integrasi Numeris

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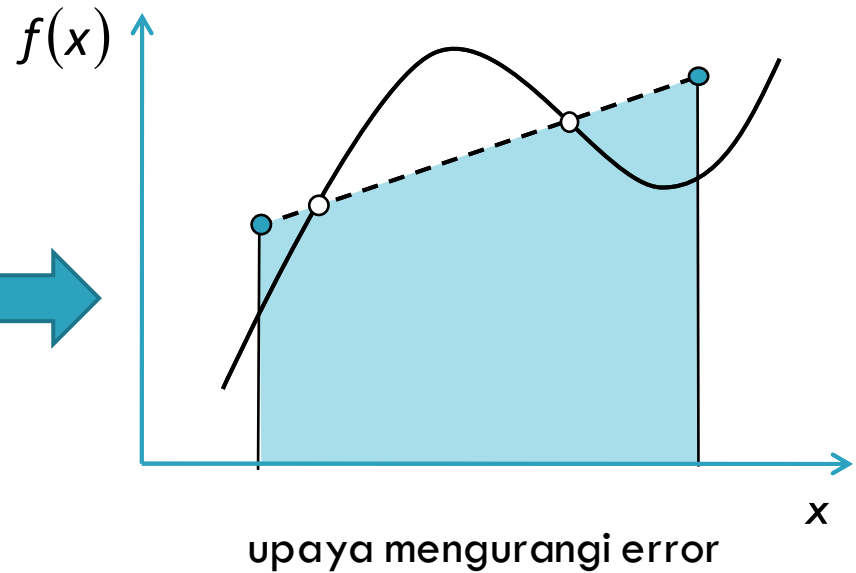
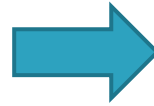
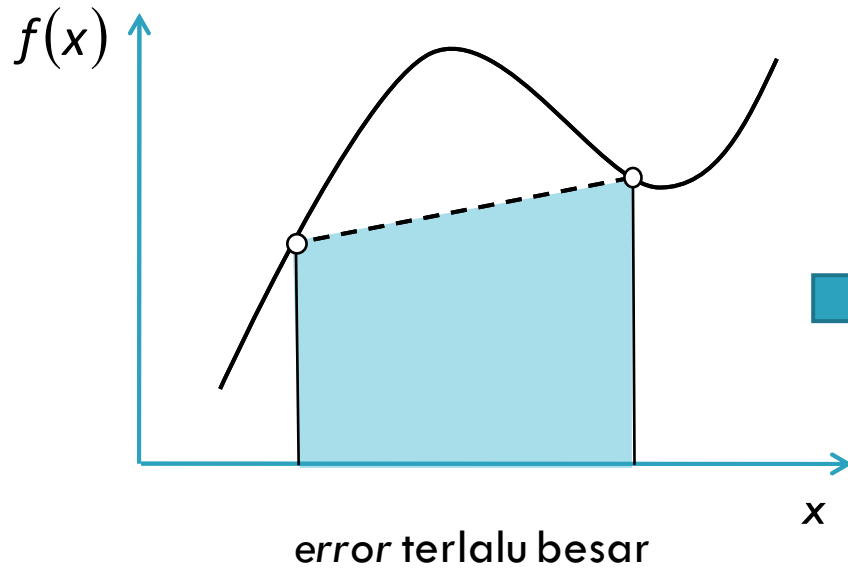
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Metode	Jumlah pias	Lebar pias
Trapeسيوم	1	
Trapeسيوم multi pias	$n > 1$	seragam atau tak-seragam
Simpson $\frac{1}{3}$	2	seragam
Simpson $\frac{1}{3}$ multi pias	genap ($2m, m = 2,3,\dots$)	seragam
Simpson $\frac{3}{8}$	3	seragam
Kuadratur Gauss	1	

Kuadratur Gauss

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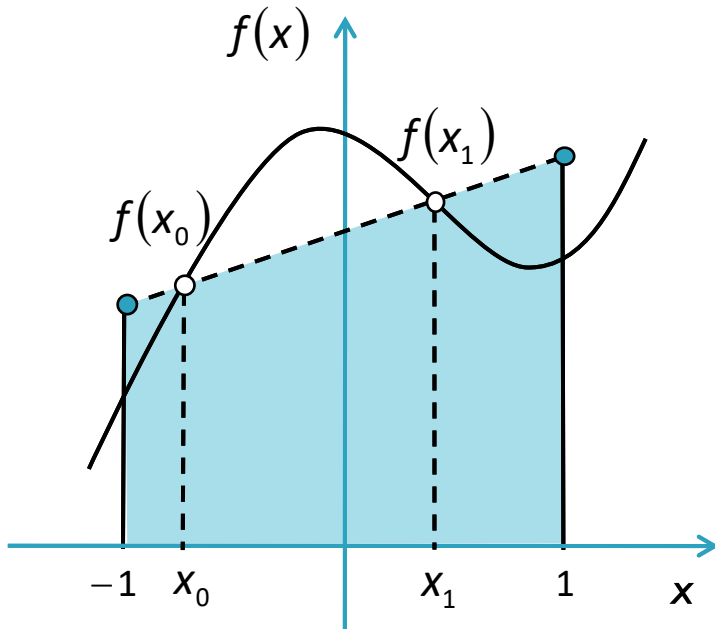
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Kuadratur Gauss

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Kuadratur Gauss 2 Titik: Gauss-Legendre

$$I = \int_{-1}^1 f(x) dx \approx c_0 f(x_0) + c_1 f(x_1)$$

c_0, c_1, x_0, x_1 : *unknowns*

$$c_0 f(x_0) + c_1 f(x_1) = \int_{-1}^1 1 dx = 2$$

$$c_0 f(x_0) + c_1 f(x_1) = \int_{-1}^1 x dx = 0$$

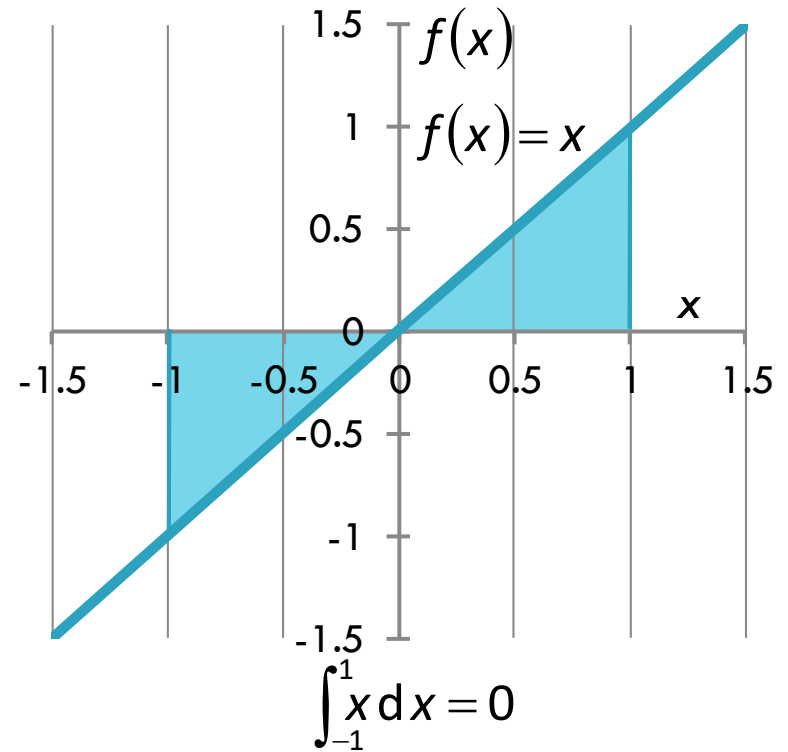
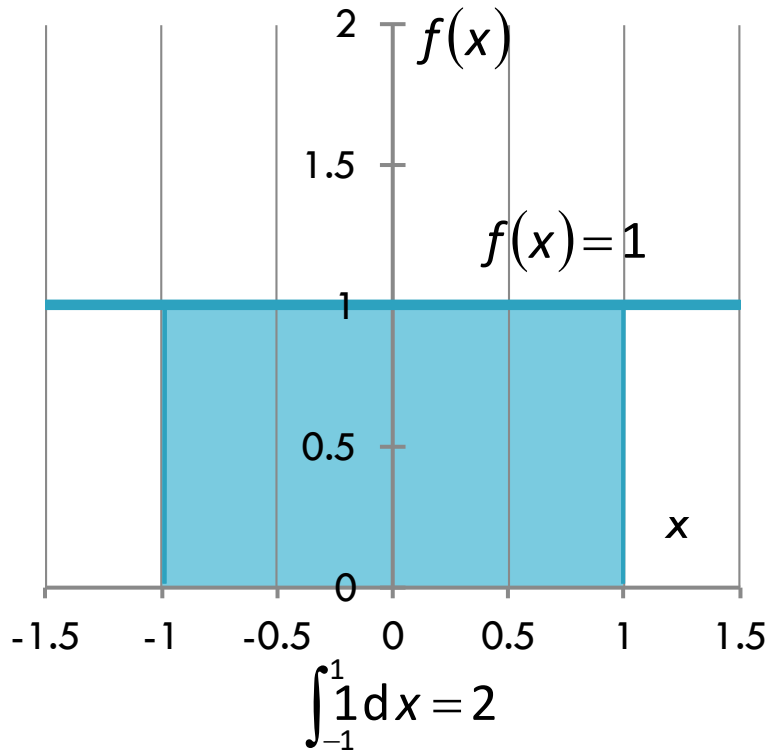
$$c_0 f(x_0) + c_1 f(x_1) = \int_{-1}^1 x^2 dx = 2/3$$

$$c_0 f(x_0) + c_1 f(x_1) = \int_{-1}^1 x^3 dx = 0$$

Kuadratur Gauss

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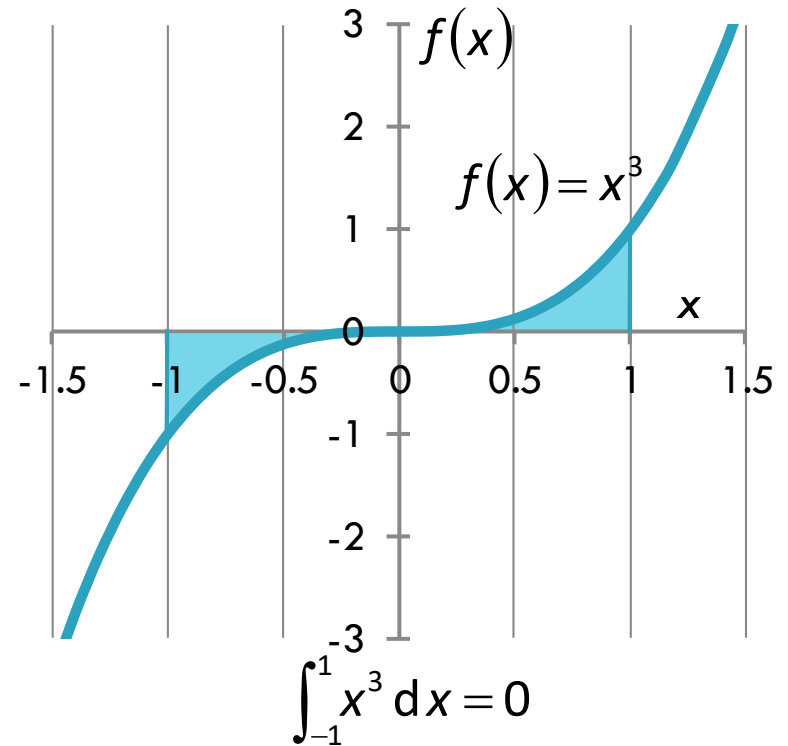
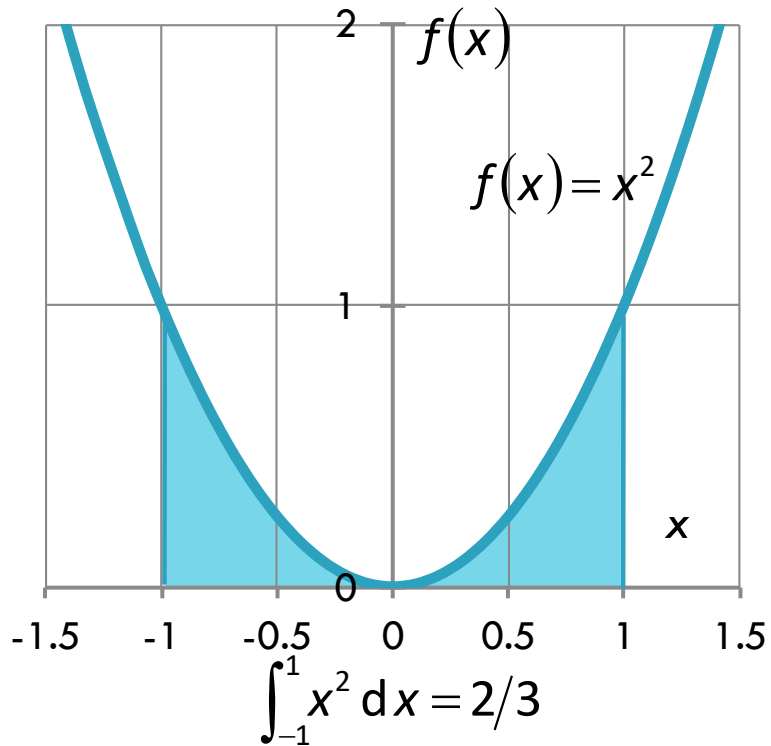
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Kuadratur Gauss

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Kuadratur Gauss

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$$c_0 f(x_0) + c_1 f(x_1) = \int_{-1}^1 1 dx = 2$$

$$c_0 f(x_0) + c_1 f(x_1) = \int_{-1}^1 x dx = 2$$

$$c_0 f(x_0) + c_1 f(x_1) = \int_{-1}^1 x^2 dx = 2/3$$

$$c_0 f(x_0) + c_1 f(x_1) = \int_{-1}^1 x^3 dx = 0$$



$$c_0 = c_1 = 1$$

$$x_0 = -1/\sqrt{3}$$

$$x_1 = 1/\sqrt{3}$$



$$I \approx c_0 f(x_0) + c_1 f(x_1)$$



$$I \approx f(-1/\sqrt{3}) + f(1/\sqrt{3})$$

Kuadratur Gauss

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Untuk batas integrasi dari a ke b :

- diambil asumsi suatu variabel x_d yang dapat dihubungkan dengan variabel asli x dalam suatu relasi linear

$$x = a_0 + a_1 x_d$$

- jika batas bawah, $x = a$, berkaitan dengan $x_d = -1 \Rightarrow a = a_0 + a_1(-1)$
- jika batas atas, $x = b$, berkaitan dengan $x_d = 1 \Rightarrow b = a_0 + a_1(1)$

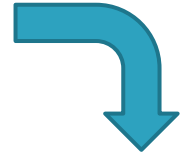
$$x = \frac{(b+a) + (b-a)x_d}{2}$$
$$dx = \frac{(b-a)}{2} dx_d$$



$$x = a_0 + a_1 x_d$$



$$a_0 = \frac{b+a}{2} \text{ dan } a_1 = \frac{b-a}{2}$$



Kuadratur Gauss

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

$$I = \int_0^{0.8} f(x) dx = 1.640533 \quad (\text{exact solution})$$

Penyelesaian dengan Metode Kuadratur Gauss:

$$x = \frac{(0.8 + 0) + (0.8 - 0)x_d}{2} = 0.4 + 0.4x_d$$

$$dx = \frac{0.8 - 0}{2} dx_d = 0.4 dx_d$$

Kuadratur Gauss

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$$\int_0^{0.8} (0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5) dx =$$

$$\int_{-1}^1 \left\{ \begin{array}{l} 0.2 + 25(0.4 + 0.4x_d) - 200(0.4 + 0.4x_d)^2 + 675(0.4 + 0.4x_d)^3 - \\ 900(0.4 + 0.4x_d)^4 + 400(0.4 + 0.4x_d)^5 \end{array} \right\} 0.4 dx_d =$$

$$f(x_d = -1/\sqrt{3}) = 0.516741$$

$$f(x_d = 1/\sqrt{3}) = 1.305837$$



$$I \approx \int_0^{0.8} f(x) dx$$

$$= 0.516741 + 1.305837$$

$$= 1.822578$$

Sekian