UNIVERSITAS GADJAH MADA
FAKULTAS TEKNIK
DEPARTEMEN TEKNIK SIPIL DAN LINGKUNGAN

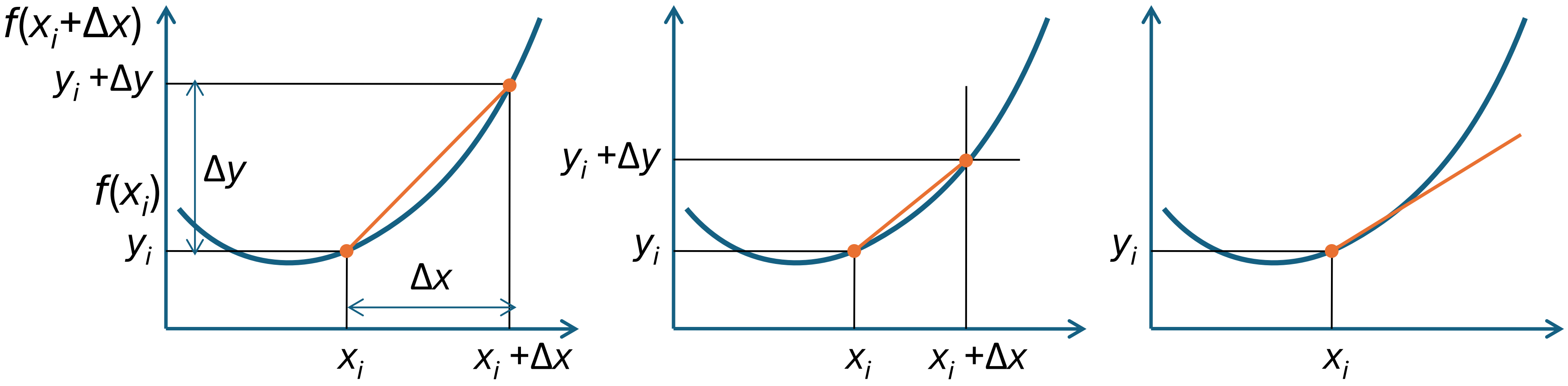
INTEGRASI NUMERIK

Numerical Differentiation and Integration

Integrasi Numerik

- Acuan
  - Chapra, S.C., Canale R.P., 1990, *Numerical Methods for Engineers*, 2nd Ed., McGraw-Hill Book Co., New York.
    - Chapter 15 dan 16, hlm. 459-523.

Diferensial, Derivatif



(a) (b) (c)

$$\frac{\Delta y}{\Delta x} = \frac{f(x_i + \Delta x) - f(x_i)}{\Delta x}$$

difference approximation

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x_i + \Delta x) - f(x_i)}{\Delta x}$$

Diferensial, Derivatif

$$\frac{\Delta y}{\Delta x} = \frac{f(x_i + \Delta x) - f(x_i)}{\Delta x}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x_i + \Delta x) - f(x_i)}{\Delta x}$$

pendekatan beda (hingga)
difference approximation

derivatif

Deret Taylor

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x - a)^n$$

$$\frac{dy}{dx} = y' = f'(x)$$

derivatif = laju perubahan y terhadap x

Diferensial, Derivatif

Deret Taylor $$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Deret Taylor adalah sebuah pendekatan terhadap fungsi $f(x)$ di suatu titik $x = a$.

$$\Rightarrow f(x) = \frac{f(a)}{0!}(x-a)^0 + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

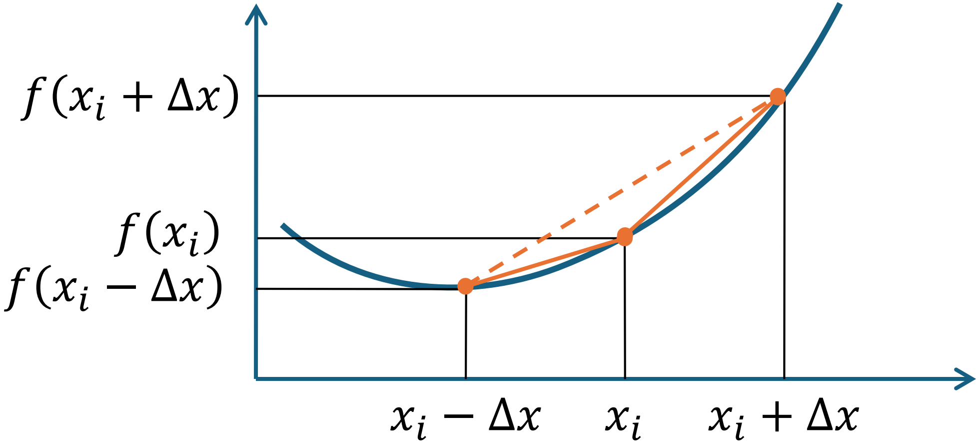
$$\Rightarrow f(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \frac{1}{6}f'''(a)(x-a)^3 + \cdots$$

$$\Rightarrow f(x) = f(a) + f'(a)(x-a) + O(x)$$

$$\Rightarrow f'(x=a) \approx \frac{f(x)-f(a)}{(x-a)}$$

Diferensial, Derivatif

pendekatan beda (hingga)
difference approximation



diferensi maju (*forward difference*)

$$\frac{dy}{dx} \approx \frac{\Delta y}{\Delta x} = \frac{f(x_i + \Delta x) - f(x_i)}{\Delta x}$$

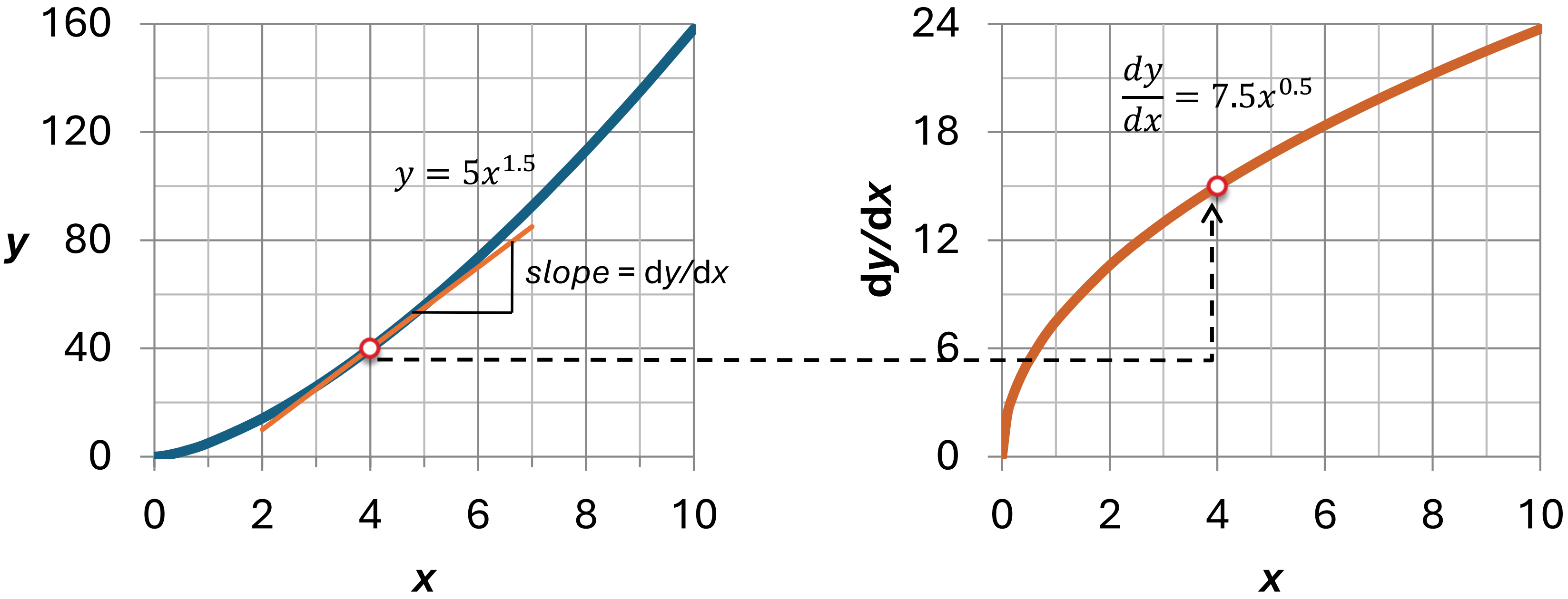
diferensi mundur (*backward difference*)

$$\frac{dy}{dx} \approx \frac{\Delta y}{\Delta x} = \frac{f(x_i) - f(x_i - \Delta x)}{\Delta x}$$

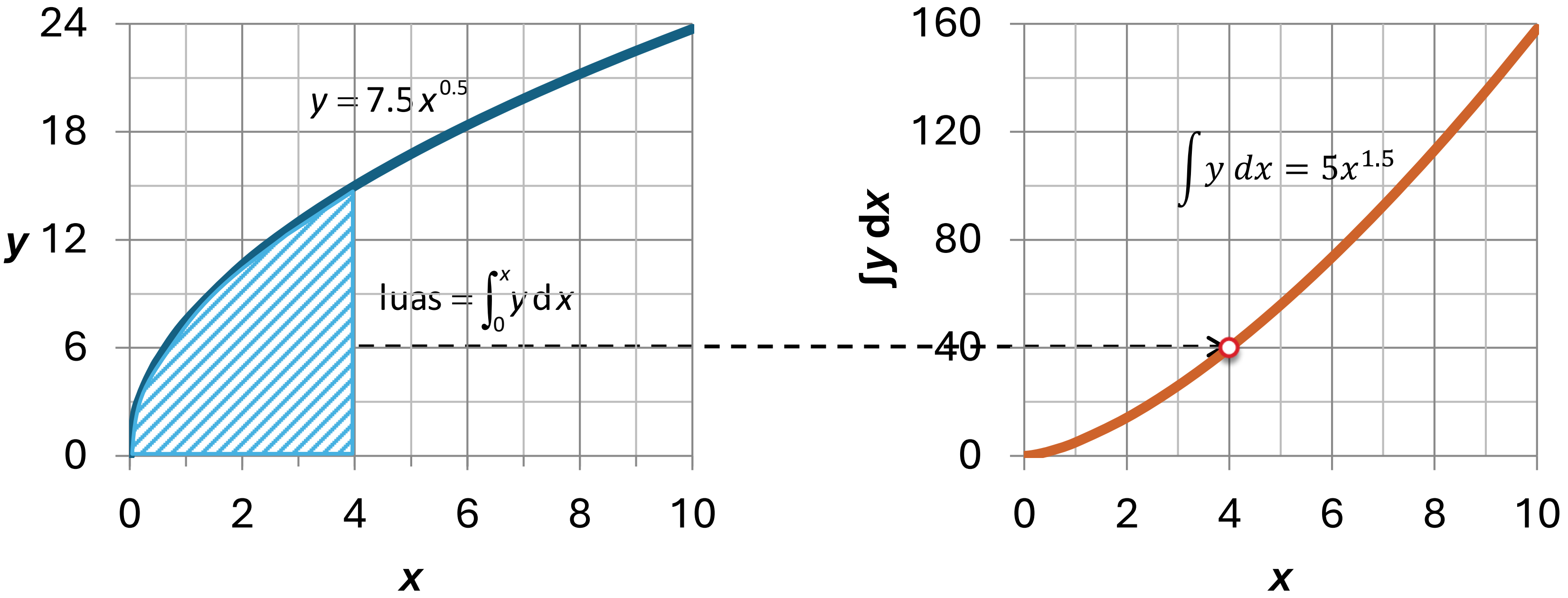
diferensi tengah (*central difference*)

$$\frac{dy}{dx} \approx \frac{\Delta y}{\Delta x} = \frac{f(x_i + \Delta x) - f(x_i - \Delta x)}{2\Delta x}$$

Diferensial



Integral



- “kebalikan” proses men-diferensial-kan adalah meng-integral-kan
- integrasi ⟩⟨ diferensiasi

Fungsi

- Fungsi-fungsi yang di-diferensial-kan atau di-integral-kan dapat berupa
  - fungsi kontinu sederhana: polinomial, eksponensial, trigonometri;
  - fungsi kontinu kompleks yang tidak memungkinkan didiferensialkan atau dintegralkan secara langsung;
  - fungsi yang nilai-nilainya disajikan dalam bentuk tabel (tabulasi data x vs $f(x)$).

Cara mencari nilai integral

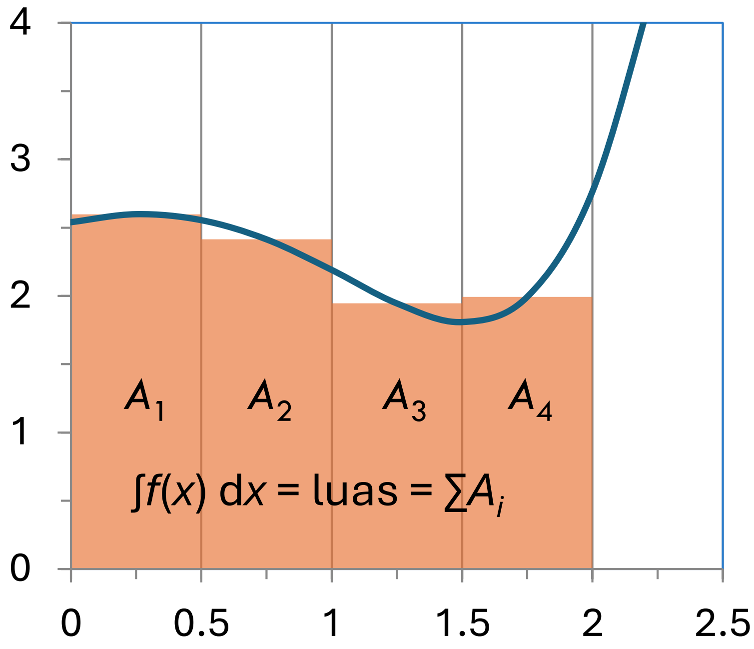
1

$$\int_0^2 \frac{2 + \cos\left(1 + x^{3/2}\right)}{\sqrt{1 + 0.5 \sin x}} e^{0.5x} dx$$

2

x	***f*(*x*)**
0.25	2.599
0.75	2.414
1.25	1.945
1.75	1.993

3



Derivatif

$u = f(x)$ dan $v = f(x)$

$$y = u^n \Rightarrow \frac{dy}{dx} = n u^{n-1} \frac{du}{dx}$$

$$y = uv \Rightarrow \frac{dy}{dx} = v \frac{du}{dx} + u \frac{dv}{dx}$$

$$y = \frac{u}{v} \Rightarrow \frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

$$\frac{d}{dx} \sin x = \cos x$$

$$\frac{d}{dx} \cos x = -\sin x$$

$$\frac{d}{dx} \tan x = \sec^2 x$$

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

$$\frac{d}{dx} e^x = e^x$$

$$\frac{d}{dx} \cot x = -\csc^2 x$$

$$\frac{d}{dx} \sec x = \sec x \tan x$$

$$\frac{d}{dx} \csc x = -\csc x \cot x$$

$$\frac{d}{dx} \log_a x = \frac{1}{x \ln a}$$

$$\frac{d}{dx} a^x = a^x \ln a$$

Integral

$$\int u dv = uv - \int v \, \mathrm{d} u$$

$$\int u^n dv = \frac{u^{n+1}}{n+1} + C \qquad n \neq -1$$

$$\int a^{bx} dx = \frac{a^{bx}}{b \ln a} + C \qquad a > 0, \quad a \neq 1$$

$$\int \frac{dx}{x} = \ln|x| + C$$

$$\int \sin(ax + b) \, dx = -\frac{1}{a} \cos(ax + b) + C$$

$$\int \cos(ax + b) \, dx = \frac{1}{a} \sin(ax + b) + C$$

$$\int \ln|x| \, dx = x \ln|x| - x + C$$

$$\int e^{ax} dx = \frac{e^{ax}}{a} + C$$

$$\int x e^{ax} dx = \frac{e^{ax}}{a^2}(ax - 1) + C$$

$$\int \frac{dx}{a + bx^2} = \frac{1}{\sqrt{ab}} \tan^{-1} \frac{\sqrt{ab}}{a} x + C$$

Metode Integrasi Newton-Cotes

Metode Trapesium

Metode Simpson

Metode Kuadratur Gauss

Persamaan Newton-Cotes

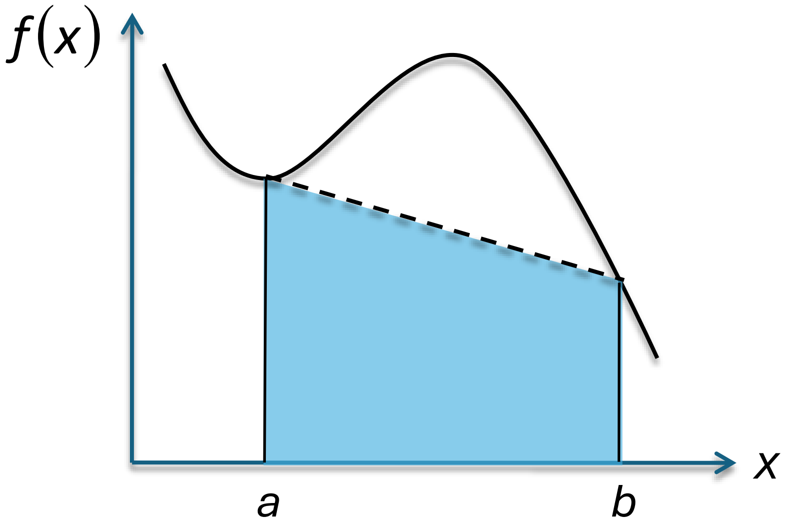
- Strategi
  - mengganti fungsi kompleks dan rumit atau tabulasi data dengan **fungsi pendekatan** yang mudah untuk diintegralkan

$$I = \int_a^b f(x)\,dx = \int_a^b f_n(x)\,dx$$

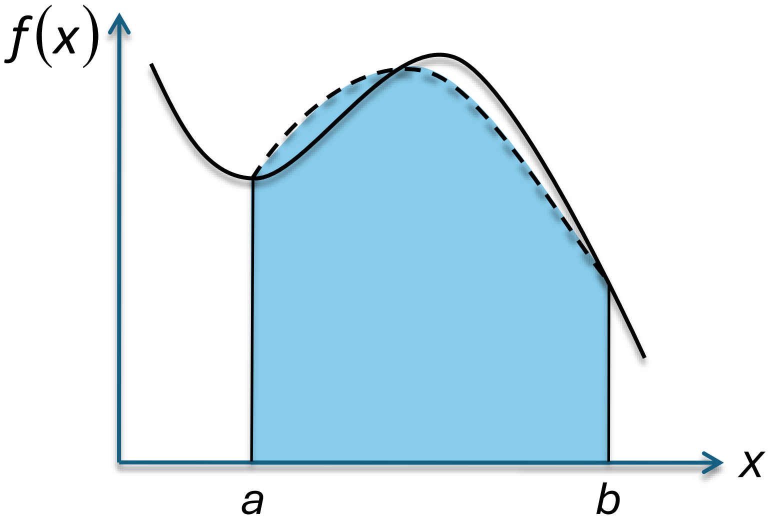
$$f_n(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-1} x^{n-1} + a_n x^n$$

polinomial orde n

Persamaan Newton-Cotes

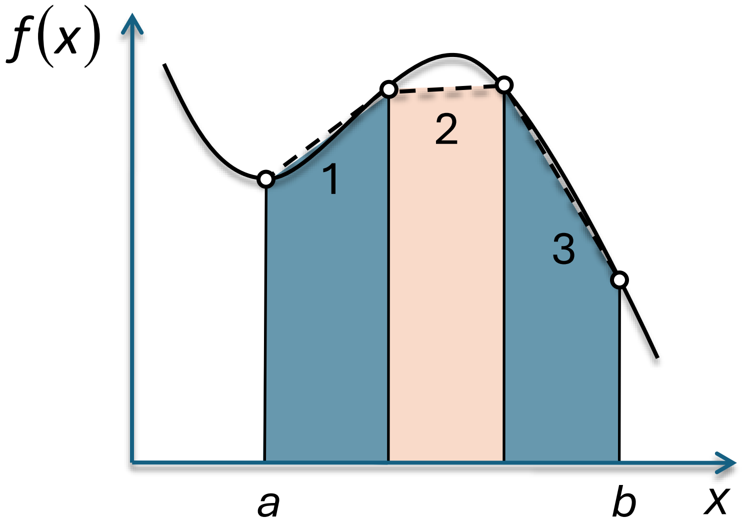


Garis lurus (polinomial orde 1) sebagai fungsi pendekatan.



Kurva parabola (polinomial orde 2) sebagai fungsi pendekatan.

Persamaan Newton-Cotes



Garis lurus (polinomial orde 1) sebagai fungsi pendekatan.

Fungsi yang diintegralkan didekati dengan 3 buah garis lurus (polinomial orde 1).
Dapat pula dipakai beberapa kurva polinomial orde yang lebih tinggi.

Metode Trapesium

- Fungsi pendekatan untuk menghitung integral adalah polinomial orde 1

$$I = \int_a^b f(x)\,\mathrm{d}\,x = \int_a^b f_1(x)\,\mathrm{d}\,x$$

- Sebuah garis lurus dapat dinyatakan dengan persamaan

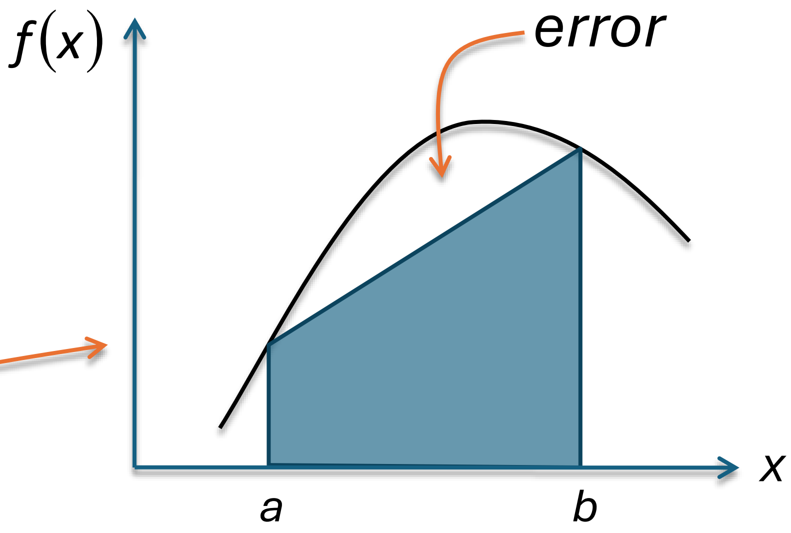
$$f_1(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$

Metode Trapesium

$$I \cong \int_a^b \left\{ f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \right\} dx$$

$$\cong (b - a)\, \frac{f(a) + f(b)}{2}$$

Metode Trapesium



Metode Trapesium

Penyelesaian eksak

$$I = \int_0^{0.8} (0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5)\ \mathrm{d}\,x$$

$$= \left(0.2x + 12.5x^2 - \frac{200}{3}x^3 + \frac{675}{4}x^4 - 180x^5 + \frac{400}{6}x^6\right)\Bigg|_0^{0.8} = 1.640533$$

Metode Trapesium

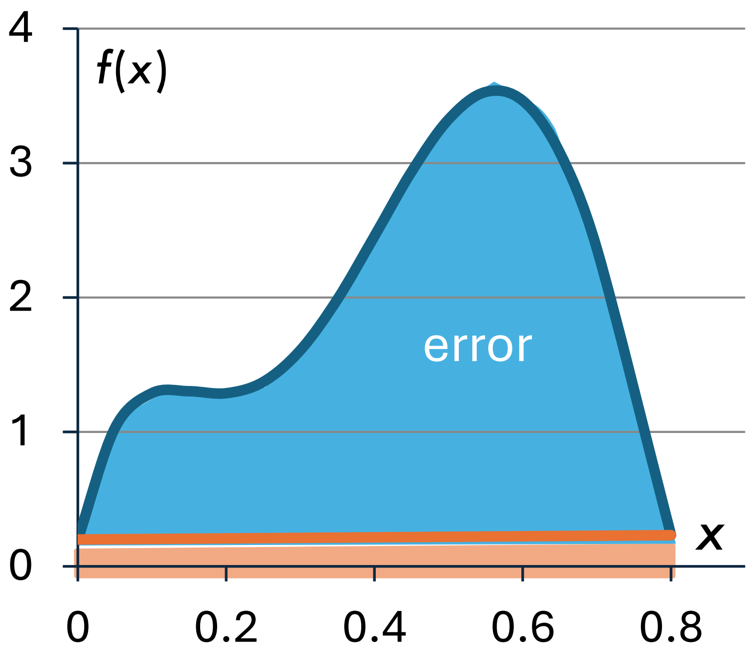
$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

$$f(0) = 0.2 \ \text{ dan } \ f(0.8) = 0.232$$

[*error*]

$$I = (0.8 - 0)\frac{0.2 + 0.232}{2} = 0.1728 \Rightarrow E_t = 1.640533 - 0.1728 = 1.467733 \quad (\approx 89\%)$$

Metode Trapesium

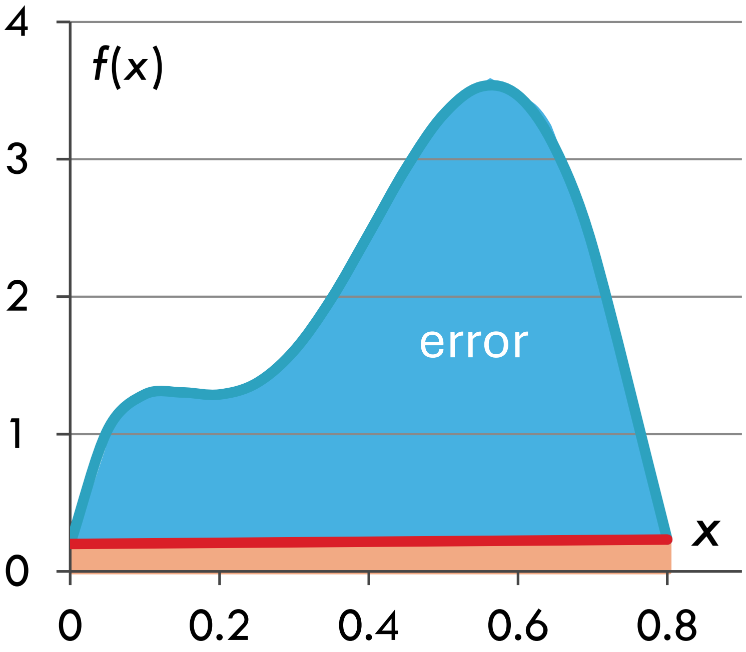


- *Error* atau kesalahan
  - bentuk trapesium untuk menghitung nilai integral mengabaikan sejumlah besar porsi daerah di bawah kurva
- Kuantifikasi *error* metode trapesium

$$E_t = -\frac{1}{12} f''(\xi)\ (b-a)^3$$

ξ adalah titik di antara a dan b

Metode Trapesium



$$f''(x) = -400 + 4050x - 10800x^2 + 8000x^3$$

nilai rerata derivatif kedua

$$\overline{f''(x)} = \frac{\int_0^{0.8} (-400 + 4050x - 10800x^2 + 8000x^3)dx}{(0.8 - 0)}$$

error

$$E_a = -\frac{1}{12}(-0.60)\ (0.8 - 0)^3 = 2.56$$

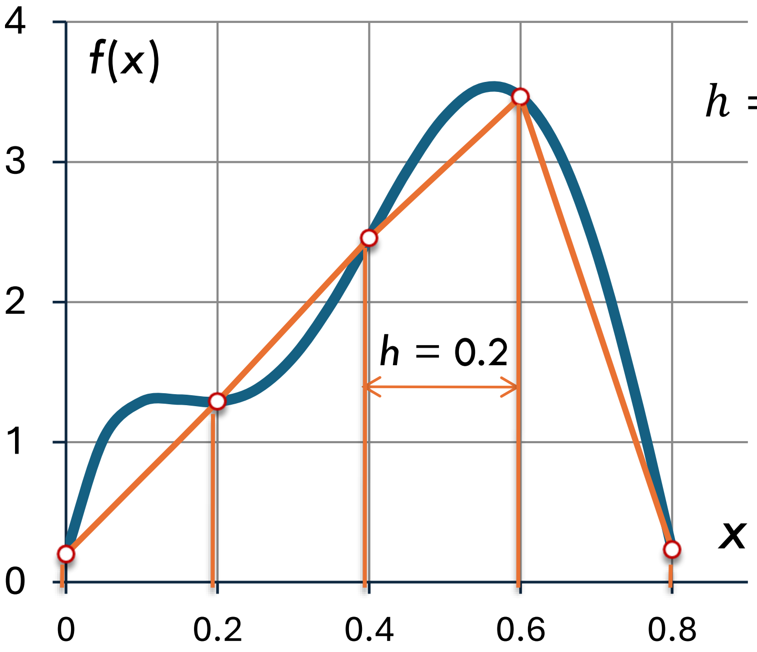
order of magnitude nilai *error* ini sama dengan *order of magnitude* nilai *error* terhadap nilai penyelesaian eksak dan keduanya sama tanda (sama-sama positif)

Trapesium multi pias

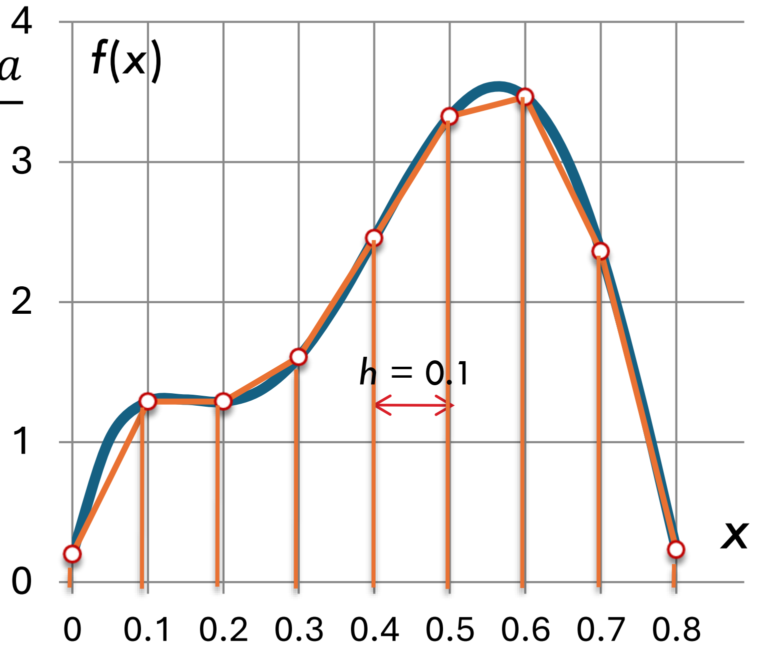
- Peningkatan akurasi
  - selang ab dibagi menjadi sejumlah n pias yang memiliki lebar seragam h

$$h = \frac{b - a}{n}$$

Trapesium multi pias



$$h = \frac{b-a}{n}$$



Trapesium multi pias

$$h = \frac{b-a}{n} \Rightarrow \text{Jika } a = x_0 \text{ dan } b = x_n$$

$$I = \int_{x_0}^{x_1} f(x)\,\mathrm{d}x + \int_{x_1}^{x_2} f(x)\,\mathrm{d}x + \ldots + \int_{x_{n-1}}^{x_n} f(x)\,\mathrm{d}x$$

$$I \approx h\frac{f(x_0)+f(x_1)}{2} + h\frac{f(x_1)+f(x_2)}{2} + \ldots + h\frac{f(x_{n-1})+f(x_n)}{2}$$

$$I \approx \frac{h}{2}\left[f(x_0) + \left\{2\sum_{i=1}^{n-1} f(x_i)\right\} + f(x_n)\right] \Rightarrow I \approx \underbrace{(b-a)}_{\text{lebar}} \underbrace{\frac{f(x_0) + \left\{2\sum_{i=1}^{n-1} f(x_i)\right\} + f(x_n)}{2n}}_{\text{tinggi rerata}}$$

Trapesium multi pias

Error = jumlah *error* di tiap pias

$$E_t = -\frac{(b-a)^3}{12n^3}\sum_{i=1}^{n} f''(\xi_i)$$

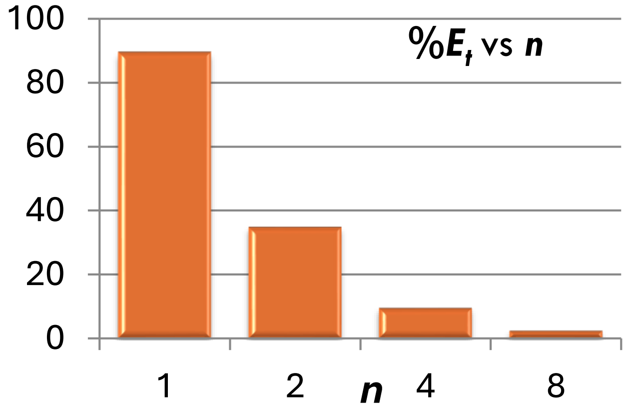
$$\frac{1}{n}\sum_{i=1}^{n} f''(\xi_i) = \bar{f}'' \quad \Rightarrow \quad E_t \approx E_a = -\frac{(b-a)^3}{12n^2}\bar{f}''$$

tiap kelipatan jumlah pias, *error* mengecil dengan faktor kuadrat peningkatan jumlah pias

Trapesium multi pias

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5 \Rightarrow I \approx \int_0^{0.8} f_1(x)\,\mathrm{d}x$$

$$I = \int_0^{0.8} f(x)dx = 1.640533$$ *(exact solution)*



(the trapezoidal rule)

n	h	I	E_t	$\%E_t$	E_a
1	0.8	0.1728	1.4677	89%	2.56
2	0.4	1.0688	0.5717	35%	0.64
4	0.2	1.4848	0.1557	9%	0.16
8	0.1	1.6008	0.0397	2%	0.04

Metode Simpson

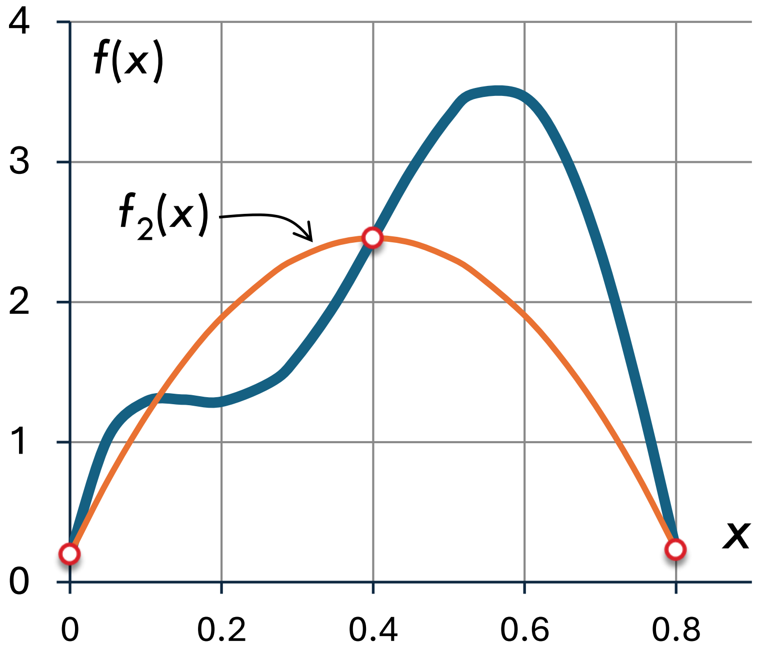
The trapezoidal rule

- Fungsi pendekatan polinomial orde 1
  - Peningkatan ketelitian dilakukan dengan meningkatkan jumlah pias

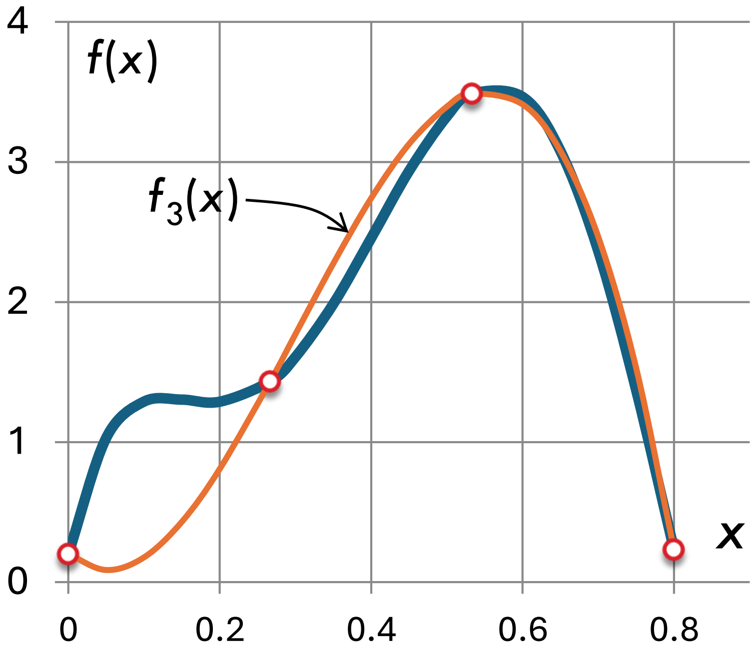
Simpson's rules

- Fungsi pendekatan
  - polinomial orde 2 → Simpson 1/3
  - polinomial orde 3 → Simpson 3/8

Metode Simpson



Simpson ⅓



Simpson ⅜

Metode Simpson

- Polinomial orde 2 atau 3
  - dicari dengan Metode Newton atau Lagrange (lihat materi tentang *curve fitting*)

Simpson ⅓

$$I = \int_a^b f(x)\,\mathrm{d}\,x \approx \int_a^b f_2(x)\,\mathrm{d}\,x$$

Jika $a = x_0$ dan $b = x_n$, dan $f_2(x)$ diperoleh dengan Metode Lagrange

$$I \approx \int_a^b \left\{ \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2) \right\} dx$$

$$I \approx \frac{h}{3}\{f(x_0) + 4f(x_1) + f(x_2)\}$$ atau $$I \approx (b-a)\frac{f(x_0) + 4f(x_1) + f(x_2)}{6}$$

$$h = \frac{b-a}{2}$$

$$E_t = -\frac{(b-a)^5}{180n^4} f^4(\xi)$$

Simpson ⅓ multi pias

$$I \approx \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \ldots + \int_{x_{n-2}}^{x_n} f(x)dx$$

$$I \approx 2h\frac{f(x_0) + 4f(x_1) + f(x_2)}{6} + 2h\frac{f(x_2) + 4f(x_3) + f(x_4)}{6} + \ldots + 2h\frac{f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)}{6}$$

$$I \approx (b-a)\frac{f(x_0) + 4\sum_{i=1,3,5}^{n-1} f(x_i) + 2\sum_{i=2,4,6}^{n-2} f(x_i) + f(x_n)}{3n}$$

$\bar{f}^4$

$$E_a = -\frac{(b-a)^5}{180n^4}\bar{f}^4$$

estimasi *error*, $\bar{f}^4$, rerata derivatif ke-4

Simpson ⅜

$$I = \int_a^b f(x)\,\mathrm{d}\,x \approx \int_a^b f_3(x)\,\mathrm{d}\,x$$

Jika $a = x_0$ dan $b = x_n$, dan $f_3(x)$ diperoleh dengan Metode Lagrange

$$I \approx \int_a^b \left\{ \begin{array}{l} \dfrac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \dfrac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1) + \\ \dfrac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \dfrac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3) \end{array} \right\} dx$$

$$I \approx \frac{3h}{8}\{f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\}$$ atau $$I \approx \underbrace{(b-a)}_{\text{lebar}} \underbrace{\frac{f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)}{8}}_{\text{tinggi rerata}}$$

$$h = \frac{b-a}{3}$$

Simpson ⅜

Error

$$E_t = -\frac{3}{80} h^5 \; f^4(\xi) \quad \text{atau} \quad E_t = -\frac{(b-a)^5}{6480} f^4(\xi)$$

Simpson ⅓ dan ⅜

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

$$I = \int_0^{0.8} f(x)dx = 1.640533 \qquad \textit{(exact solution)}$$

Metode	I	E_t
Simpson ⅓ (n = 2)	1.367467	0.273067 (27%)
Simpson ⅜ (n = 3)	1.51917	0.121363 (7%)
Simpson ⅓ (n = 4)	1.623467	0.017067 (1%)

Pias tak seragam: Metode Trapesium

i	x_i	$f(x_i)$	I
0	0	0.2	
1	0.12	1.309729	0.090584
2	0.22	1.305241	0.130749
3	0.32	1.743393	0.152432
4	0.36	2.074903	0.076366
5	0.4	2.456	0.090618
6	0.44	2.842985	0.10598
7	0.54	3.507297	0.317514
8	0.64	3.181929	0.334461
9	0.7	2.363	0.166348
10	0.8	0.232	0.12975
			1.594801

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

I dihitung dengan Metode Trapesium di tiap pias

$$I = h_1 \frac{f(x_1) + f(x_0)}{2} + h_2 \frac{f(x_2) + f(x_1)}{2} + \ldots + h_n \frac{f(x_n) + f(x_{n-1})}{2}$$

$$\int_0^{0.8} f(x)\,\mathrm{d}x = 1.594801$$

Pias tak seragam: Metode Simpson

i	x_i	$f(x_i)$	I
0	0	0.2	trapesium
1	0.12	1.309729	
2	0.22	1.305241	
3	0.32	1.743393	
4	0.36	2.074903	
5	0.4	2.456	
6	0.44	2.842985	
7	0.54	3.507297	
8	0.64	3.181929	
9	0.7	2.363	
10	0.8	0.232	

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

I dihitung dengan Metode Simpson ⅓ dan Simpson ⅜

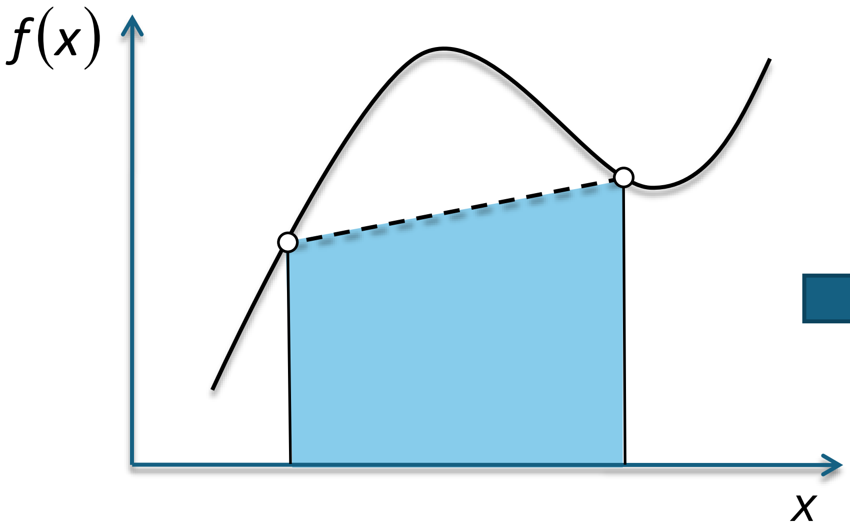
?

PR, dikumpulkan minggu depan

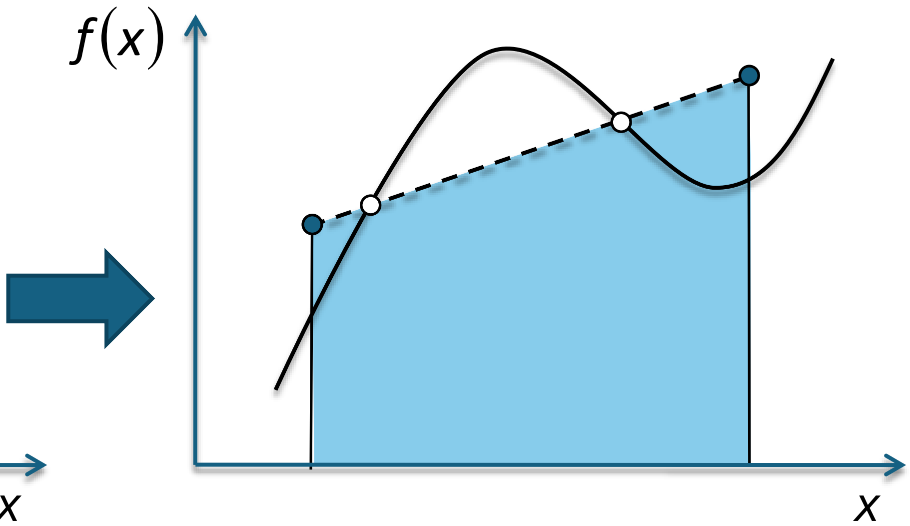
Metode Integrasi Numerik

Metode	Jumlah pias	Lebar pias
Trapesium	1	
Trapesium multi pias	n > 1	seragam atau tak-seragam
Simpson ⅓	2	seragam
Simpson ⅓ multi pias	genap (2m, m = 2,3,...)	seragam
Simpson ⅜	3	seragam
Kuadratur Gauss	1	

Kuadratur Gauss

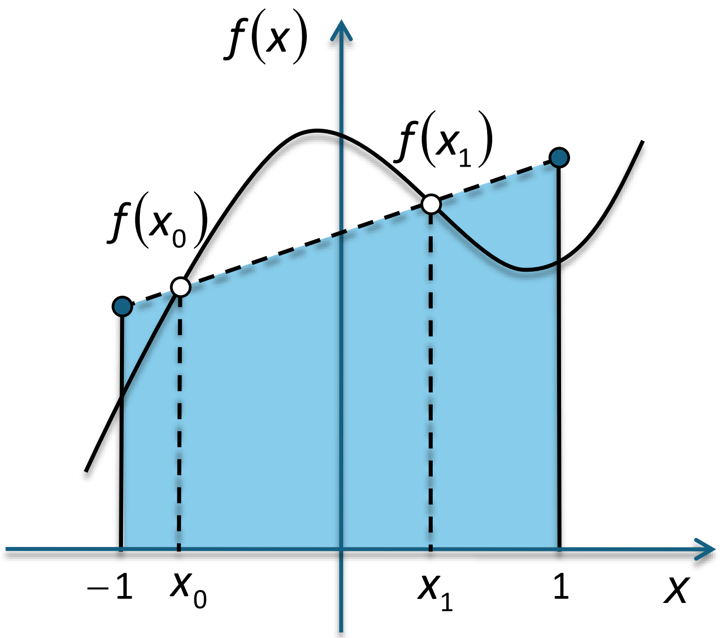


error terlalu besar



upaya mengurangi error

Kuadratur Gauss



Kuadratur Gauss 2 Titik: Gauss-Legendre

$$I = \int_{-1}^{1} f(x)dx \approx c_0 f(x_0) + c_1 f(x_1)$$

c_0, c_1, x_0, x_1 adalah *unkonwns*

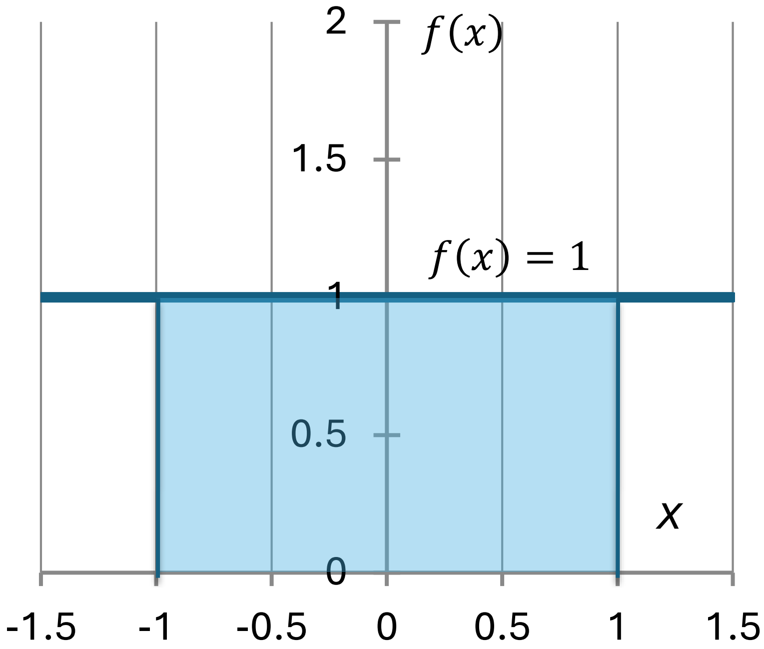
$$c_0 f(x_0) + c_1 f(x_1) = \int_{-1}^{1} 1dx = 2$$

$$c_0 f(x_0) + c_1 f(x_1) = \int_{-1}^{1} xdx = 0$$

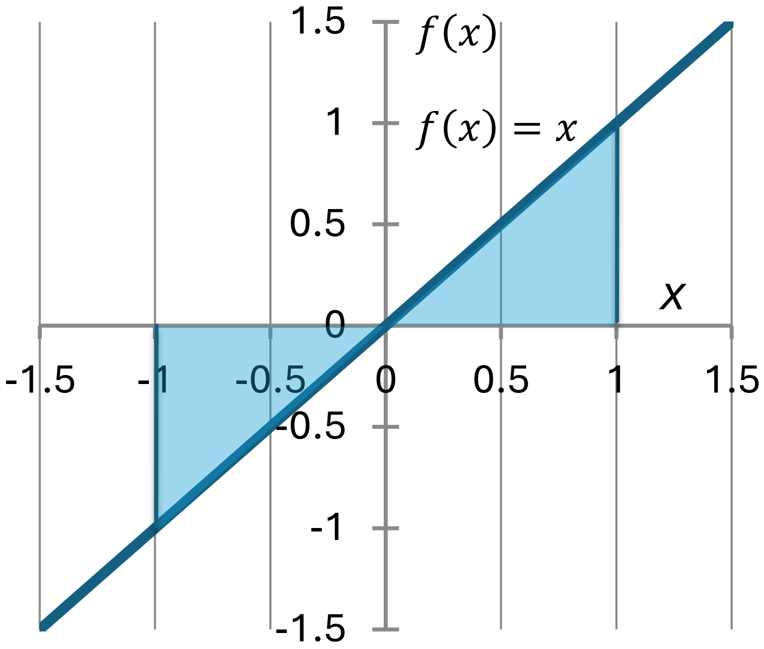
$$c_0 f(x_0) + c_1 f(x_1) = \int_{-1}^{1} x^2dx = 2/3$$

$$c_0 f(x_0) + c_1 f(x_1) = \int_{-1}^{1} x^3dx = 0$$

Kuadratur Gauss

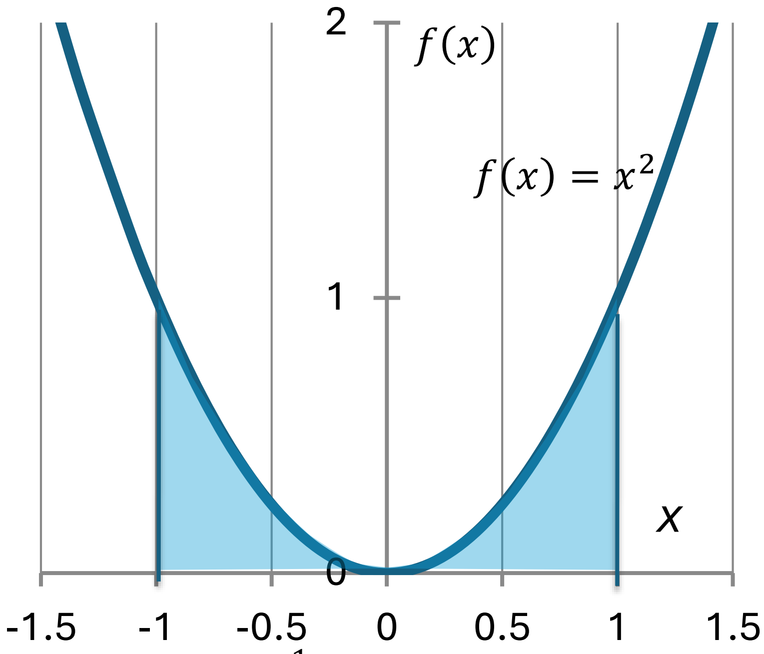


$$\int_{-1}^{1} 1\,dx = 2$$

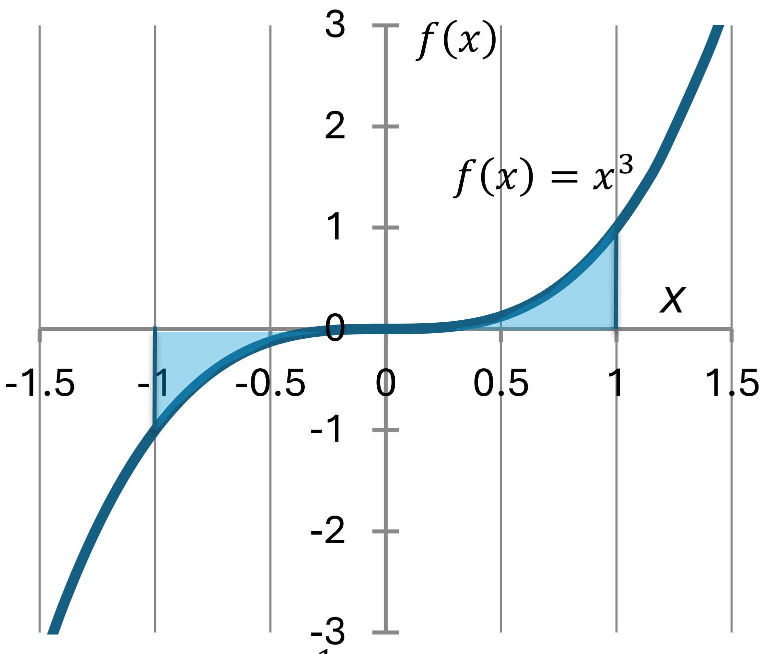


$$\int_{-1}^{1} x\,dx = 0$$

Kuadratur Gauss



$$\int_{-1}^{1} x^2 dx = 2/3$$



$$\int_{-1}^{1} x^3 dx = 0$$

Kuadratur Gauss

$$c_0 f(x_0) + c_1 f(x_1) = \int_{-1}^{1} 1\,dx = 2$$

$$c_0 f(x_0) + c_1 f(x_1) = \int_{-1}^{1} x\,dx = 2$$

$$c_0 f(x_0) + c_1 f(x_1) = \int_{-1}^{1} x^2 dx = 2/3$$

$$c_0 f(x_0) + c_1 f(x_1) = \int_{-1}^{1} x^3 dx = 0$$

$$c_0 = c_1 = 1$$

$$x_0 = -1/\sqrt{3}$$

$$x_1 = 1/\sqrt{3}$$

$$I \approx c_0 f(x_0) + c_1 f(x_1)$$

$$I \approx f\left(-1/\sqrt{3}\right) + f\left(1/\sqrt{3}\right)$$

Kuadratur Gauss

Untuk batas integrasi dari *a* ke *b*

- diambil asumsi suatu variabel x_d yang dapat dihubungkan dengan variabel asli *x* dalam suatu relasi linear

$$x = a_0 + a_1 x_d$$

- jika batas bawah, $x = a$, berkaitan dengan $x_d = -1 \Longrightarrow b = a_0 + a_1(-1)$
- jika batas atas, $x = b$, berkaitan dengan $x_d = 1 \Longrightarrow b = a_0 + a_1(1)$

$$a_0 = \frac{b+a}{2} \quad \text{dan} \quad a_1 = \frac{b-a}{2}$$

$$x = a_0 + a_1 x_d$$

$$x = \frac{(b+a) + (b-a)\ x_d}{2}$$

$$dx = \frac{(b-a)}{2} dx_d$$

Kuadratur Gauss

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

$$I = \int_0^{0.8} f(x)\,\mathrm{d}\,x = 1.640533 \quad (\textit{exact solution})$$

Penyelesaian menggunakan Metode Kuadratur Gauss

$$x = \frac{(0.8 + 0) + (0.8 - 0)x_d}{2} = 0.4 + 0.4x_d$$

$$dx = \frac{0.8 - 0}{2} dx_d = 0.4dx_d$$

Kuadratur Gauss

$$I = \int_0^{0.8} (0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5)dx$$

$$I \cong \int_{-1}^{1} \left\{ \begin{matrix} 0.2 + 25(0.4 + 0.4x_d) - 200(0.4 + 0.4x_d)^2 + 675(0.4 + 0.4x_d)^3 - \\ 900(0.4 + 0.4x_d)^4 + 400(0.4 + 0.4x_d)^5 \end{matrix} \right\} 0.4dx_d$$

$$f\left(x_d = -1/\sqrt{3}\right) = 0.516741$$

$$f\left(x_d = 1/\sqrt{3}\right) = 1.305837$$

$$I = \int_0^{0.8} f(x)dx \cong f\left(x_d = -1/\sqrt{3}\right) + f\left(1/\sqrt{3}\right)$$

$$I \cong 0.516741 + 1.305837$$

$$I \cong 1.822578$$

Sekian