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Numerical Methods

Linear Algebraic Equations

Linear algebraic equations

- Reference
  - Chapra, S.C., Canale, R.P., 2015, *Numerical Methods for Engineers*, 7th Ed., McGraw-Hill Book Co., New York
    - Part Three: Chapters 9 to12 (pp 231 to 343)

Linear algebraic equations

a system of linear algebraic equations

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = c_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = c_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = c_n$$

the a's are contant coefficients, the c's are constants, and n is the number of equations

need to solve these equations simultaneously, that is finding x's that satisfy every single equation in the system

The methods

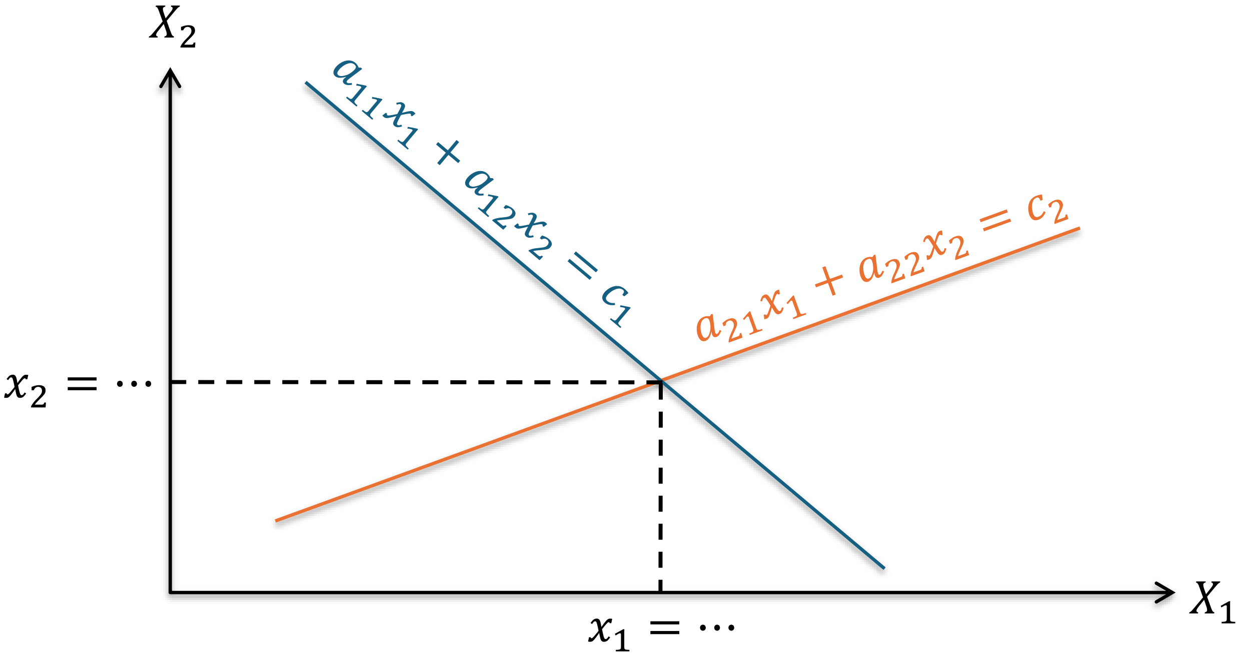
Small sets of simultaneous eqs.

- The graphical method
- Cramer’s rule
- Elimination of unknowns

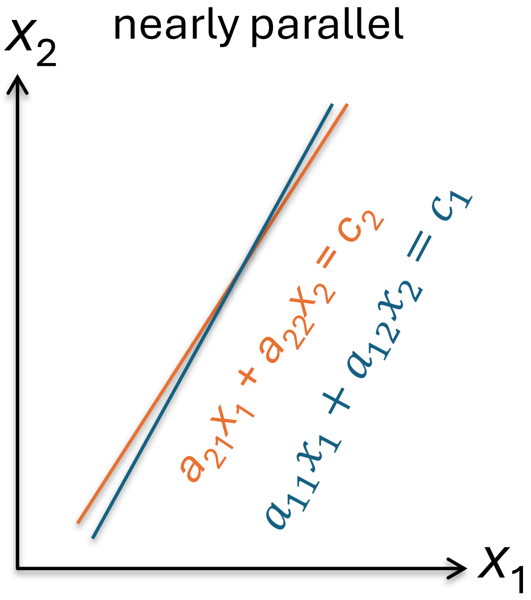
Large sets of simulatneous eqs.

- Naïve Gauss elimination
- Gauss-Jordan
- *LU* decomposition
- The matrix inverse
- Jacobi
- Gauss-Seidel
- Successive over-/under-relaxation
- Special matrices

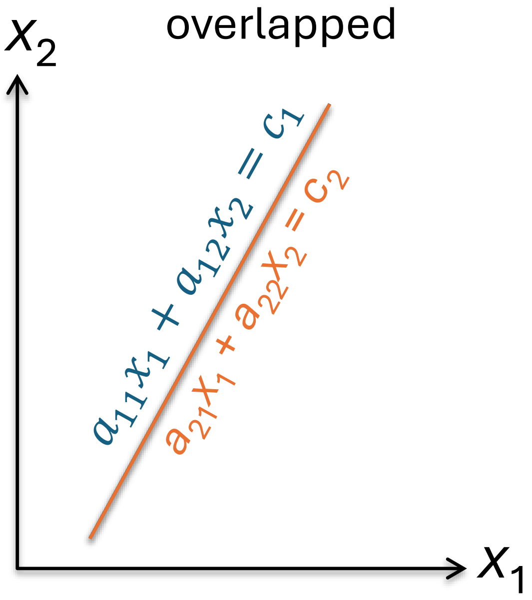
The graphical method



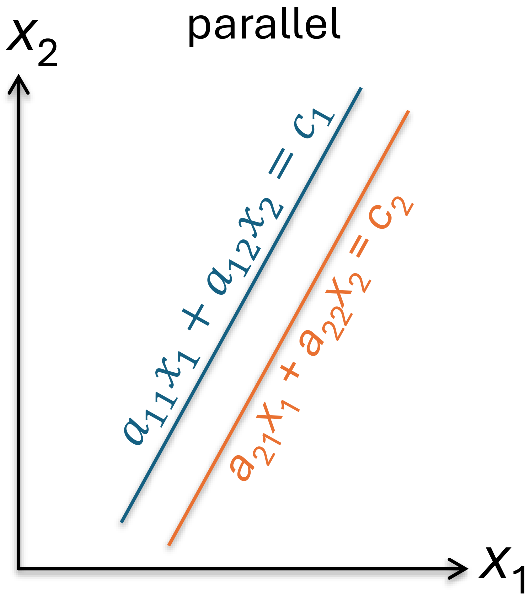
The graphical method



ill-conditioned system:
the slopes are so close that the point of intersection is difficult to detect visually



singular system:
infinite solutions



singular system:
no solution

Cramer's rule

- The rule states that each unknown in a system of linear albebraic equations may be expressed as a fraction of two deteminants with denominator D (determinant of the equations) and with the numerator obtained from D by replacing the column of coefficient of the unknown in question by the constants $c_1, c_2, \dots, c_n$

- Example

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = c_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = c_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = c_3$$

Cramer's rule

$$[A] = \boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad D = \det \boldsymbol{A} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$x_1 = \frac{\begin{vmatrix} c_1 & a_{12} & a_{13} \\ c_2 & a_{22} & a_{23} \\ c_3 & a_{32} & a_{33} \end{vmatrix}}{D} \qquad x_2 = \frac{\begin{vmatrix} a_{11} & c_1 & a_{13} \\ a_{21} & c_2 & a_{23} \\ a_{31} & c_3 & a_{33} \end{vmatrix}}{D} \qquad x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & c_1 \\ a_{21} & a_{22} & c_2 \\ a_{31} & a_{32} & c_3 \end{vmatrix}}{D}$$

Cramer’s rule

- How to find the determinant of a square 2-by-2 matrix?

$$[A] = \boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$D = \det \boldsymbol{A} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Cramer's rule

- How to find the determinant of a square 3-by-3 matrix?

$$[B] = \boldsymbol{B} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$D = \det \boldsymbol{B} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Cramer's rule

- Example

$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$
$$0.1x_1 + 7x_2 - 0.3x_3 = -19.3$$
$$0.3x_1 - 0.2x_2 + 10x_3 = 71.4$$

$$\boldsymbol{A} \cdot \boldsymbol{X} = \boldsymbol{C}$$

$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0.1 & 7 & -0.3 \\ 0.3 & -0.2 & 10 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.3 \\ 71.4 \end{Bmatrix}$$

$$\det \boldsymbol{A} = 3\{7 \times 10 - (-0.3) \times (-0.2)\} + 0.1\{0.1 \times 10 - (-0.3) \times 0.3\} - 0.2\{0.1 \times (-0.2) - 7 \times 0.3\}$$

$$\det \boldsymbol{A} = 210.353$$

Cramer's rule

$$A_1 = \begin{bmatrix} 7.85 & -0.1 & -0.2 \\ -19.3 & 7 & -0.3 \\ 71.4 & -0.2 & 10 \end{bmatrix} \quad A_2 = \begin{bmatrix} 3 & 7.85 & -0.2 \\ 0.1 & -19.3 & -0.3 \\ 0.3 & 71.4 & 10 \end{bmatrix} \quad A_3 = \begin{bmatrix} 3 & -0.1 & 7.85 \\ 0.1 & 7 & -19.3 \\ 0.3 & -0.2 & 71.4 \end{bmatrix}$$

$$\det A_1 = |A_1| = 631.059 \qquad \det A_2 = |A_2| = -525.8825 \qquad \det A_3 = |A_3| = 1472.471$$

$$x_1 = \frac{\det A_1}{\det A} = \frac{631.058}{210.353} = 3$$

$$x_2 = \frac{\det A_2}{\det A} = \frac{-525.8825}{210.353} = -2.5 \qquad x_3 = \frac{\det A_3}{\det A} = \frac{1472.471}{210.353} = 7$$

Elimination of unknowns

- The elimination of unknowns by combining equations is an algebraic approach that can be illustrated for a set of two equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 = c_1 \quad &\Longrightarrow \quad a_{21}(a_{11}x_1 + a_{12}x_2 = c_1) \quad \Longrightarrow \quad a_{21}a_{11}x_1 + a_{21}a_{12}x_2 = a_{21}c_1 \\ a_{21}x_1 + a_{22}x_2 = c_2 \quad &\Longrightarrow \quad a_{11}(a_{21}x_1 + a_{22}x_2 = c_2) \quad \Longrightarrow \quad \underline{a_{11}a_{21}x_1 + a_{11}a_{22}x_2 = a_{11}c_2} \; - \end{aligned}$$

$$a_{21}a_{12}x_2 - a_{11}a_{22}x_2 = a_{21}c_1 - a_{11}c_2$$

$$x_2 = \frac{a_{21}c_1 - a_{11}c_2}{a_{21}a_{12} - a_{11}a_{22}}$$

$$x_1 = \frac{c_1 - a_{12}x_2}{a_{11}} = \frac{c_1}{a_{11}} - \frac{a_{12}}{a_{11}}\left(\frac{a_{21}c_1 - a_{11}c_2}{a_{21}a_{12} - a_{11}a_{22}}\right)$$

Elimination of unknowns

- Example

$$\begin{aligned} 3x_1 + 2x_2 = 18 \quad &\Rightarrow \quad -(3x_1 + 2x_2 = 18) \quad \Rightarrow \quad -3x_1 - 2x_2 = -18 \\ -x_1 + 2x_2 = 2 \quad &\Rightarrow \quad 3(-x_1 + 2x_2 = 2) \quad \Rightarrow \quad -3x_1 + 6x_2 = 6 \end{aligned}$$

$$\overline{\qquad\qquad\qquad\qquad} \; -$$

$$-8x_2 = -24$$

$$x_2 = \frac{-24}{-8} = 3$$

$$x_1 = \frac{18 - 2x_2}{3} = \frac{18 - 2(3)}{3} = \frac{12}{3} = 4$$

Naïve Gauss elimination

- Strategy
  - Forward elimination of unknowns
  - Backward substitution

- Example

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = c_1 \quad (1)$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = c_2 \quad (2)$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = c_3 \quad (3)$$

Naïve Gauss elimination

- Forward elimination #1
  - eliminate x_1 from eqs. (2) and (3) by using eq. (1) as the pivot

pivot coefficient (a_{11})

pivot equation

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = c_1$$

$$\left(a_{22} - \frac{a_{21}}{a_{11}}a_{12}\right)x_2 + \left(a_{23} - \frac{a_{21}}{a_{11}}a_{13}\right)x_3 = c_2 - \frac{a_{21}}{a_{11}}c_1$$

$$\left(a_{32} - \frac{a_{31}}{a_{11}}a_{12}\right)x_2 + \left(a_{33} - \frac{a_{31}}{a_{11}}a_{13}\right)x_3 = c_3 - \frac{a_{31}}{a_{11}}c_1$$

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = c_1 \quad (1)$$

$$a'_{22}x_2 + a'_{23}x_3 = c'_2 \quad (2')$$

$$a'_{32}x_2 + a'_{33}x_3 = c'_3 \quad (3')$$

Naïve Gauss elimination

- Forward elimination #2
  - eliminate x_2 from eq. (3) by using eq. $(2')$ as the pivot

pivot coefficient *pivot equation*

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = c_1$$

$$a'_{22}x_2 + a'_{23}x_3 = c'_2$$

$$\left(a'_{33} - \frac{a'_{32}}{a'_{22}}a'_{23}\right)x_3 = c'_3 - \frac{a'_{32}}{a'_{22}}c'_2$$

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = c_1 \quad (1)$$

$$a'_{22}x_2 + a'_{23}x_3 = c'_2 \quad (2')$$

$$a''_{33}x_3 = c''_3 \quad (3'')$$

Naïve Gauss elimination

- Backward substitution
  - find x_3 from eq. $(3'')$, x_2 from eq. $(2')$, and x_1 from eq. (1)

$$x_3 = \frac{c_3''}{a_{33}''} \quad \Rightarrow \quad x_2 = \frac{c_2' - a_{23}' x_3}{a_{22}'} \quad \Rightarrow \quad x_1 = \frac{c_1 - a_{12} x_2 - a_{13} x_3}{a_{11}}$$

Naïve Gauss elimination

Forward elimination

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = c_1$$
$$a'_{22}x_2 + a'_{23}x_3 + \cdots + a'_{2n}x_n = c'_2$$
$$a''_{23}x_3 + \cdots + a''_{3n}x_n = c''_3$$
$$\vdots$$
$$a_{nn}^{n-1}x_n = c_n^{n-1}$$

Backward substitution

$$x_n = \frac{c_n^{n-1}}{a_{nn}^{n-1}}$$

$$x_i = \frac{c_i^{i-1} - \sum_{j=i+1} a_{ij}^{i-1} x_j}{a_{ii}^{i-1}}, \quad i = n-1, n-2, \cdots, 1$$

Naïve Gauss elimination

- Example

$$
\begin{aligned}
(1)\quad & 3x_1 - 0.1x_2 - 0.2x_3 = 7.85 \\
(2)\quad & 0.1x_1 + 7x_2 - 0.3x_3 = -19.3 \\
(3)\quad & 0.3x_1 - 0.2x_2 + 10x_3 = 71.4
\end{aligned}
$$

Naïve Gauss elimination

- Forward elimination
  - eliminate x_1 from eqs. (2) and (3) by using eq. (1) as the pivot

$$(1) \quad 3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$

$$(2') \quad 0x_1 + 7.0033x_2 - 0.2933x_3 = -19.5617$$

$$(3') \quad 0x_1 - 0.19x_2 + 10.02x_3 = 70.615$$

  - eliminate x_2 from eq. (3') by using eq. $(2')$ as the pivot

$$(1) \quad 3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$

$$(2') \quad 0x_1 + 7.0033x_2 - 0.2933x_3 = -19.5617$$

$$(3'') \quad 0x_1 + 0x_2 + 10.0120x_3 = 70.0843$$

Naïve Gauss elimination

- eliminate x_1 from eqs. (2) by using eq. (1) as the pivot, $(2) \longrightarrow (2')$
  - eq. (2) – ($a_{21}/a_{11} *$ eq. (1))

$$(2) \quad 0.1x_1 + 7x_2 - 0.3x_3 = -19.3$$

$$\frac{0.1}{3}(3x_1 - 0.1x_2 - 0.2x_3) = \frac{0.1}{3}7.85 \quad -$$

$$\underbrace{\left(0.1 - \frac{0.1}{3}3\right)}_{0} x_1 + \underbrace{\left(7 + \frac{0.1}{3}0.1\right)}_{7.0033} x_2 + \underbrace{\left(-0.3 + \frac{0.1}{3}0.2\right)}_{-0.2933} x_3 = \underbrace{-19.3 - \frac{0.1}{3}7.85}_{-19.5617}$$

$$(2') \Rightarrow \quad 0x_1 + 7.0033x_2 - 0.2933x_3 = -19.5617$$

Naïve Gauss elimination

- Back substitution
  - find x_3 from eq. $(3'')$

$$x_3 = \frac{70.0843}{10.0120} = 7$$

  - substitute x_3 to eq. $(2')$ to get x_2

$$x_2 = \frac{-19.5617 + 0.2933(7)}{7.0033} = -2.5$$

  - substitute x_3 and x_2 to eq. (1) to get x_1

$$x_1 = \frac{7.85 + 0.2 \times 7 + 0.1(-2.5)}{3} = 3$$

Naïve Gauss elimination

- Pitfalls of the naïve Gauss elimination
  - it is possible that division by zero can occur during both the forward elimination and the back substitution phases
  - problem of round-off errors can become particularly important when large numbers of equations are to be solved
    - this is due to the fact that every result is dependent on previous results
    - an error in the early steps will tend to propagate, that is, it will cause errors in subsequent steps
  - ill-conditioned systems, where small changes in coefficients result in large changes in the solution

Naïve Gauss elimination

- Techniques for improving solutions
  - determine the largest available coefficient in the column below the pivot element
  - switch the rows so that the largest element is the pivot element
  - ➔ rearranging the equations such that the pivot equation is the one that gives the largest pivot coefficient

Gauss-Jordan

- Gauss-Jordan method is a variation of Gauss elimination
  - when an unknown is eliminated in the Gauss-Jordan method, it is eliminated from all other equations rather than just the subsequent ones
  - all rows are normalized by dividing them by their pivot elements
  - the elimination step results in an identity matrix rather than a triangular matrix
- Example

$$
\begin{array}{ll}
(1) & 3x_1 - 0.1x_2 - 0.2x_3 = 7.85 \\
(2) & 0.1x_1 + 7x_2 - 0.3x_3 = -19.3 \\
(3) & 0.3x_1 - 0.2x_2 + 10x_3 = 71.4
\end{array}
$$

Gauss-Jordan

$$\left[\begin{array}{ccc|c} 3 & -0.1 & -0.2 & 7.85 \\ 0.1 & 7 & -0.3 & -19.3 \\ 0.3 & -0.1 & 10 & 71.4 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 3/3 & -0.1/3 & -0.2/3 & 7.85/3 \\ 0.1 & 7 & -0.3 & -19.3 \\ 0.3 & -0.1 & 10 & 71.4 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -0.0333 & -0.0667 & 2.6167 \\ 0.1 & 7 & -0.3 & -19.3 \\ 0.3 & -0.1 & 10 & 71.4 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -0.0333 & -0.0667 & 2.6167 \\ 0 & 7.0033 & -0.2933 & -19.5617 \\ 0 & -0.1900 & 10.0200 & 70.6150 \end{array}\right]$$

Gauss-Jordan

$$\left[\begin{array}{ccc|c} 1 & -0.0333 & -0.0667 & 2.6167 \\ 0 & 7.0033 & -0.2933 & -19.5617 \\ 0 & -0.1900 & 10.0200 & 70.6150 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -0.0333 & -0.0667 & 2.6167 \\ 0/7.0033 & 7.0033/7.0033 & -0.2933/7.0033 & -19.5617/7.0033 \\ 0 & -0.1900 & 10.0200 & 70.6150 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -0.0333 & -0.0667 & 2.6167 \\ 0 & 1 & -0.0419 & -2.7931 \\ 0 & -0.1900 & 10.0200 & 70.6150 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -0.0681 & 2.5236 \\ 0 & 1 & -0.0419 & -2.7931 \\ 0 & 0 & 10.0120 & 70.0843 \end{array}\right]$$

Gauss-Jordan

$$\left[\begin{array}{ccc|c} 1 & 0 & -0.0681 & 2.5236 \\ 0 & 1 & -0.0419 & -2.7931 \\ 0/10.0120 & 0/10.0120 & 10.0120/10.0120 & 70.0843/10.0120 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -0.0681 & 2.5236 \\ 0 & 1 & -0.0419 & -2.7931 \\ 0 & 0 & 1 & 7 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2.5 \\ 0 & 0 & 1 & 7 \end{array}\right] \Rightarrow \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 3 \\ -2.5 \\ 7 \end{Bmatrix}$$

Gauss-Jordan

- It has more operations than Gauss elimination by 50%
- It has the same pitfalls, i.e.
  - division by zero
  - large round-off errors

LU decomposition

- Gauss elimination → *LU* decomposition

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{[A]} \longrightarrow \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ f_{21} & 1 & 0 \\ f_{31} & f_{32} & 1 \end{bmatrix}}_{\{L\}} \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix}}_{[U]}$$

$$f_{21} = \frac{a_{21}}{a_{11}}$$

$$f_{31} = \frac{a_{31}}{a_{11}} \qquad f_{32} = \frac{a'_{32}}{a'_{22}}$$

LU decomposition

(1) $[L][U] = [A]$ → Construct $[U]$ —Step 1

→ Construct $[L]$ —Step 2

(2) $[L]\{D\} = \{C\}$ → Get $\{D\}$ —Step 3

(3) $[U]\{X\} = \{D\}$ → Obtain $\{X\}$ —Step 4

Procedure

- Construct $[U]$ —Step 1
- Construct $[L]$ —Step 2
- Get $\{D\}$ —Step 3
- Obtain $\{X\}$ —Step 4

LU decomposition

- Steps 1: construct $[U]$ following Gauss elimination

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \longrightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & a'_{32} & a'_{33} \end{bmatrix} \longrightarrow \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix}}_{[U]}$$

- Step 2: contruct $[L]$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \longrightarrow \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ f_{21} & 1 & 0 \\ f_{31} & f_{32} & 1 \end{bmatrix}}_{[L]}$$

$$f_{21} = \frac{a_{21}}{a_{11}}$$

$$f_{31} = \frac{a_{31}}{a_{11}} \qquad f_{32} = \frac{a'_{32}}{a'_{22}}$$

LU decomposition

- Step 3: get $\{D\}$

$[L]\{D\} = \{C\}$

$$\begin{bmatrix} 1 & 0 & 0 \\ f_{21} & 1 & 0 \\ f_{31} & f_{32} & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix}$$

$$d_1 = c_1$$

$$f_{21} d_1 + d_2 = c_2$$
$$\Leftrightarrow d_2 = c_2 - f_{21} d_1$$

$$f_{31} d_1 + f_{32} d_2 + d_3 = c_3$$
$$\Leftrightarrow d_3 = c_3 - f_{31} d_1 - f_{32} d_2$$

- Step 4: obtain $\{X\}$

$[U]\{X\} = \{D\}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix}$$

$$a''_{33} x_3 = d_3$$
x_3 $\Leftrightarrow x_3 = d_3 / a''_{33}$

$$a'_{22} x_2 + a'_{23} x_3 = d_2$$
x_2 $\Leftrightarrow x_2 = (d_2 - a'_{23} x_3) / a'_{22}$

$$a_{11_{x_1}} + a_{12} x_2 + a_{13} x_3 = d_1$$
x_1 $\Leftrightarrow x_1 = (d_1 - a_{12} x_2 - a_{13} x_3) / a_{11}$

LU decomposition

- Example

$$(1)\quad 3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$
$$(2)\quad 0.1x_1 + 7x_2 - 0.3x_3 = -19.3$$
$$(3)\quad 0.3x_1 - 0.2x_2 + 10x_3 = 71.4$$

$$\underbrace{\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0.1 & 7 & -0.3 \\ 0.3 & -0.2 & 10 \end{bmatrix}}_{[A]} \underbrace{\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}}_{\{X\}} = \underbrace{\begin{Bmatrix} 7.85 \\ -19.3 \\ 71.4 \end{Bmatrix}}_{\{C\}}$$

LU decomposition

$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0.1 & 7 & -0.3 \\ 0.3 & -0.2 & 10 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & -0.1 & -0.2 \\ 0 & 7.0033 & -0.2933 \\ 0 & -0.19 & 10.0200 \end{bmatrix} \Rightarrow [U] = \begin{bmatrix} 3 & -0.1 & -0.2 \\ 0 & 7.0033 & -0.2933 \\ 0 & 0 & 10.0120 \end{bmatrix}$$

$$f_{21} = \frac{a_{21}}{a_{11}} = \frac{0.1}{3} = 0.0333$$

$$f_{31} = \frac{a_{31}}{a_{11}} = \frac{0.3}{3} = 0.1$$

$$f_{32} = \frac{a'_{32}}{a'_{22}} = \frac{-0.19}{7.0033} = -0.0271$$

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ 0.0333 & 1 & 0 \\ 0.1000 & -0.0271 & 1 \end{bmatrix}$$

LU decomposition

$$[L]\{D\} = \{C\}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.0333 & 1 & 0 \\ 0.1000 & 0.0271 & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.3 \\ 71.4 \end{Bmatrix}$$

$$d_1 = 7.85$$

$$0.0333 d_1 + d_2 = -19.3 \Leftrightarrow d_2 = -19.3 - 0.0333(7.85) = -19.5617$$

$$0.1 d_1 + 0.0271 d_2 + d_3 = 71.4$$

$$\Leftrightarrow d_3 = 71.4 - 0.1(7.85) - 0.0271(-19.5617) = 70.0843$$

LU decomposition

$$[U]\{X\} = \{D\}$$

$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0 & 7.0033 & -0.2933 \\ 0 & 0 & 10.0120 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.5617 \\ 70.0843 \end{Bmatrix}$$

$$10.0120 x_3 = 70.0843 \Leftrightarrow x_3 = \frac{70.0843}{10.0120} = 7$$

$$7.0033 x_2 - 0.2933 x_3 = -19.5617 \Leftrightarrow x_2 = \frac{-19.5617 + 0.2933(7)}{7.0033} = -2.5$$

$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$

$$\Leftrightarrow x_1 = \frac{7.85 + 0.1(-2.5) + 0.2(7)}{3} = 3$$

$$\Rightarrow \{X\} = \begin{Bmatrix} 3 \\ -2.5 \\ 7 \end{Bmatrix}$$

The matrix inverse

$$[A] \cdot \{X\} = \{C\} \Rightarrow \{X\} = [A]^{-1} \cdot \{C\}$$

$$\left[\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & a_{11}^{-1} & a_{12}^{-1} & a_{13}^{-1} \\ 0 & 1 & 0 & a_{21}^{-1} & a_{22}^{-1} & a_{23}^{-1} \\ 0 & 0 & 1 & a_{31}^{-1} & a_{32}^{-1} & a_{33}^{-1} \end{array}\right]$$

The matrix inverse

- Example

$$(1)\quad 3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$
$$(2)\quad 0.1x_1 + 7x_2 - 0.3x_3 = -19.3$$
$$(3)\quad 0.3x_1 - 0.2x_2 + 10x_3 = 71.4$$

$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0.1 & 7 & -0.3 \\ 0.3 & -0.2 & 10 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.3 \\ 71.4 \end{Bmatrix}$$

$$[A] \qquad \{X\} \qquad \{C\}$$

The matrix inverse

$$[A] = \left[\begin{array}{ccc|ccc} 3 & -0.1 & -0.2 & 1 & 0 & 0 \\ 0.1 & 7 & -0.3 & 0 & 1 & 0 \\ 0.3 & -0.2 & 10 & 0 & 0 & 1 \end{array}\right] \Rightarrow [A] = \left[\begin{array}{ccc|ccc} 1 & -0.0333 & -0.0667 & 0.3333 & 0 & 0 \\ 0.1 & 7 & -0.3 & 0 & 1 & 0 \\ 0.3 & -0.2 & 10 & 0 & 0 & 1 \end{array}\right]$$

$$\Downarrow$$

$$[A] = \left[\begin{array}{ccc|ccc} 1 & -0.0333 & -0.0667 & 0.3333 & 0 & 0 \\ 0 & 7.0033 & -0.2933 & -0.0333 & 1 & 0 \\ 0 & -0.1900 & 10.0200 & -0.0999 & 0 & 1 \end{array}\right]$$

The matrix inverse

$$[A] = \left[\begin{array}{ccc|ccc} 1 & -0.0333 & -0.0667 & 0.3333 & 0 & 0 \\ 0 & 1 & -0.0417 & -0.0047 & 0.1422 & 0 \\ 0 & -0.1900 & 10.0200 & -0.0999 & 0 & 1 \end{array}\right]$$

⇓

$$[A] = \left[\begin{array}{ccc|ccc} 1 & 0 & -0.0681 & 0.3318 & 0.0047 & 0 \\ 0 & 1 & -0.0417 & -0.0047 & 0.1422 & 0 \\ 0 & 0 & 10.0121 & -0.1009 & 0.0270 & 1 \end{array}\right]$$

The matrix inverse

$$[A] = \left[\begin{array}{ccc|ccc} 1 & 0 & -0.0681 & 0.3318 & 0.0047 & 0 \\ 0 & 1 & -0.0417 & -0.0047 & 0.1422 & 0 \\ 0 & 0 & 1 & -0.0101 & 0.0027 & 0.0999 \end{array}\right]$$

$$[A] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.3325 & 0.0049 & 0.0068 \\ 0 & 1 & 0 & -0.0052 & 0.1423 & 0.0042 \\ 0 & 0 & 1 & -0.0101 & 0.0027 & 0.0999 \end{array}\right]$$

$[A]^{-1}$

The matrix inverse

$$\{X\} = [A]^{-1}\{C\}$$

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} 0.3325 & 0.0049 & 0.0068 \\ -0.0052 & 0.1423 & 0.0042 \\ -0.0101 & 0.0027 & 0.0999 \end{bmatrix} \begin{Bmatrix} 7.85 \\ -19.3 \\ 71.4 \end{Bmatrix}$$

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 3.0004 \\ -2.4881 \\ 7.0002 \end{Bmatrix}$$

The matrix inverse

- Pitfalls
  - The matrix inverse may not exist
- MS Excel
  - =MINVERSE() cntrl+shift+enter
  - =MMULT() cntrl+shift+enter

Jacobi

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = c_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = c_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = c_3$$

initial guesses, e.g. $x_i^0 = 0$

$$x_1^0 = 0$$
$$x_2^0 = 0$$
$$x_3^0 = 0$$

continue the iteration until $x_i^{n+1} \approx x_i^n, \forall x_i$

$$x_1 = \frac{c_1 - a_{12}x_2 - a_{13}x_3}{a_{11}}$$
$$x_2 = \frac{c_2 - a_{21}x_1 - a_{23}x_3}{a_{22}}$$
$$x_3 = \frac{c_3 - a_{31}x_1 - a_{32}x_2}{a_{33}}$$

$$x_1^1 = \frac{c_1 - a_{12}x_2^0 - a_{13}x_3^0}{a_{11}}$$
$$x_2^1 = \frac{c_2 - a_{21}x_1^0 - a_{23}x_3^0}{a_{22}}$$
$$x_3^1 = \frac{c_3 - a_{31}x_1^0 - a_{32}x_2^0}{a_{33}}$$

$$x_1^{n+1} = \frac{c_1 - a_{12}x_2^n - a_{13}x_3^n}{a_{11}}$$
$$x_2^{n+1} = \frac{c_2 - a_{21}x_1^n - a_{23}x_3^n}{a_{22}}$$
$$x_3^{n+1} = \frac{c_3 - a_{31}x_1^n - a_{32}x_2^n}{a_{33}}$$

Jacobi

- Example

$$(1) \quad 3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$
$$(2) \quad 0.1x_1 + 7x_2 - 0.3x_3 = -19.3$$
$$(3) \quad 0.3x_1 - 0.2x_2 + 10x_3 = 71.4$$

Jacobi

n	x_1^n	x_2^n	x_3^n	$\|\Delta \mathbf{x}\|_{max}$
0	0	0	0	
1	2.616667	-2.75714	7.14	7.14
2	3.000762	-2.48852	7.006357	0.3840952
3	3.000806	-2.49974	7.000207	0.0112146
4	3.000022	-2.5	6.999981	0.0007839
5	2.999999	-2.5	6.999999	2.385E-05

Gauss-Seidel

$$x_1^1 = \frac{c_1 - a_{12}x_2^0 - a_{13}x_3^0}{a_{11}}$$

$$x_2^1 = \frac{c_2 - a_{21}x_1^0 - a_{23}x_3^0}{a_{22}}$$

$$x_3^1 = \frac{c_1 - a_{31}x_1^0 - a_{32}x_2^0}{a_{33}}$$

$$x_1^{n+1} = \frac{c_1 - a_{12}x_2^n - a_{13}x_3^n}{a_{11}}$$

$$x_2^{n+1} = \frac{c_2 - a_{21}x_1^{n+1} - a_{23}x_3^n}{a_{22}}$$

$$x_3^{n+1} = \frac{c_1 - a_{31}x_1^{n+1} - a_{32}x_2^{n+1}}{a_{33}}$$

continue the iteration until $x_i^{n+1} \approx x_i^n, \forall x_i$

Gauss-Seidel

- Example

$$
\begin{aligned}
&(1) \quad 3x_1 - 0.1x_2 - 0.2x_3 = 7.85 \\
&(2) \quad 0.1x_1 + 7x_2 - 0.3x_3 = -19.3 \\
&(3) \quad 0.3x_1 - 0.2x_2 + 10x_3 = 71.4
\end{aligned}
$$

Gauss-Seidel

n	x_1^n	x_2^n	x_3^n	$\|\Delta x\|_{max}$
0	0	0	0	
1	2.616667	-2.79452	7.00561	7.0056095
2	2.990557	-2.49962	7.000291	0.3738898
3	3.000032	-2.49999	6.999999	0.0094754
4	3	-2.5	7	3.155E-05
5	3	-2.5	7	3.544E-07

Successive over-relaxation

$$x_1^{n+1} = \frac{c_1 - a_{12}x_2^n - a_{13}x_3^n}{a_{11}}$$

$$x_2^{n+1} = \frac{c_2 - a_{21}[\lambda x_1^{n+1} + (1-\lambda)x_1^n] - a_{23}x_3^n}{a_{22}}$$

$$x_3^{n+1} = \frac{c_3 - a_{31}[\lambda x_1^{n+1} + (1-\lambda)x_1^n] - a_{32}[\lambda x_2^{n+1} + (1-\lambda)x_2^n]}{a_{33}}$$

$$\lambda x_i^{n+1} + (1-\lambda)x_i^n \begin{cases} 0 < \lambda < 1 & \text{under-relaxation} \\ 1 < \lambda < 2 & \text{over-relaxation} \end{cases}$$

Successive over-/under-relaxation

- Example

$$(1)\quad 3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$
$$(2)\quad 0.1x_1 + 7x_2 - 0.3x_3 = -19.3$$
$$(3)\quad 0.3x_1 - 0.2x_2 + 10x_3 = 71.4$$

Successive over-/under-relaxation

$\lambda = 1.5$

n	x_1^n	x_2^n	x_3^n	$\lvert\Delta \mathbf{x}\rvert_{max}$
0	0	0	0	
1	2.616667	-2.81321	6.937854	6.93785357
2	2.985416	-2.50509	6.997886	0.36874976
3	2.999689	-2.50019	6.99984	0.01427299
4	2.999983	-2.50001	6.999998	0.00029368
5	3	-2.5	7	1.6465E-05

Numerical Methods

Linear Algebraic Equations