

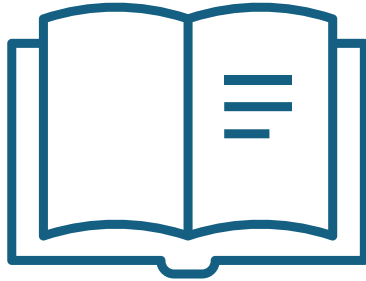


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Numerical Methods

Linear Algebraic Equations

Linear algebraic equations



■ Reference

- Chapra, S.C., Canale, R.P., 2015, *Numerical Methods for Engineers*, 7th Ed., McGraw-Hill Book Co., New York
 - Part Three: Chapters 9 to 12 (pp 231 to 343)

Linear algebraic equations

a system of linear algebraic equations

$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= c_1 \\a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= c_2 \\&\vdots \\a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= c_n\end{aligned}$$

the a 's are constant coefficients, the c 's are constants, and n is the number of equations

need to solve these equations simultaneously, that is finding x 's that satisfy every single equation in the system

The methods

Small sets of simultaneous eqs.

The graphical method

Cramer's rule

Elimination of unknowns

Large sets of simultaneous eqs.

Naïve Gauss elimination

Gauss-Jordan

LU decomposition

The matrix inverse

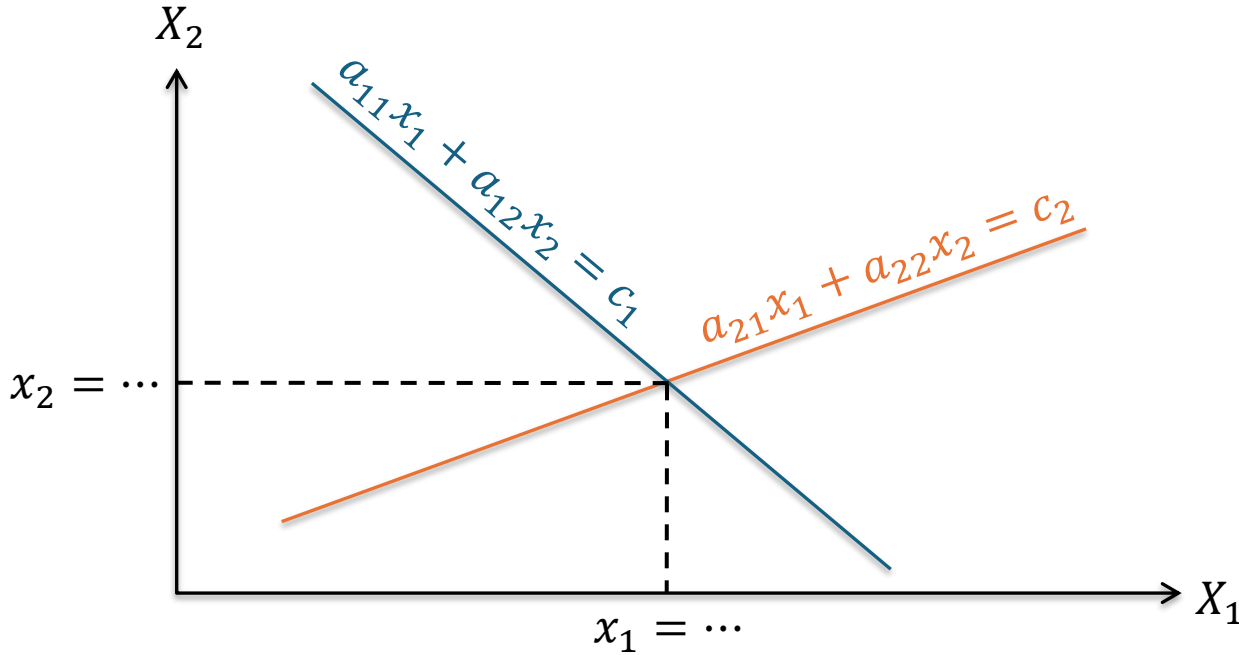
Jacobi

Gauss-Seidel

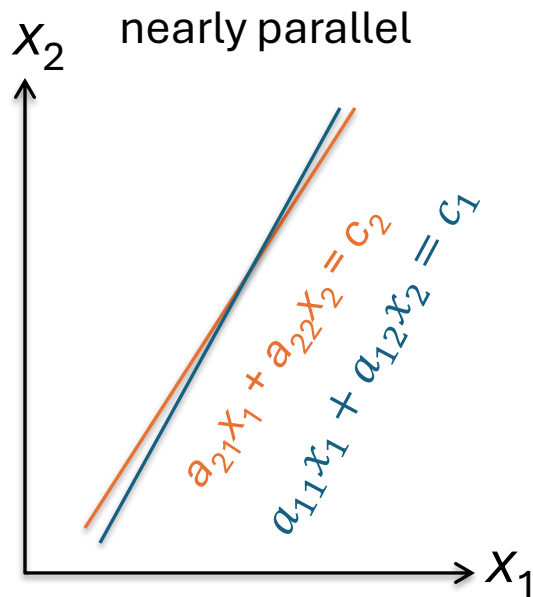
Successive under-/over-relaxation

Special matrices

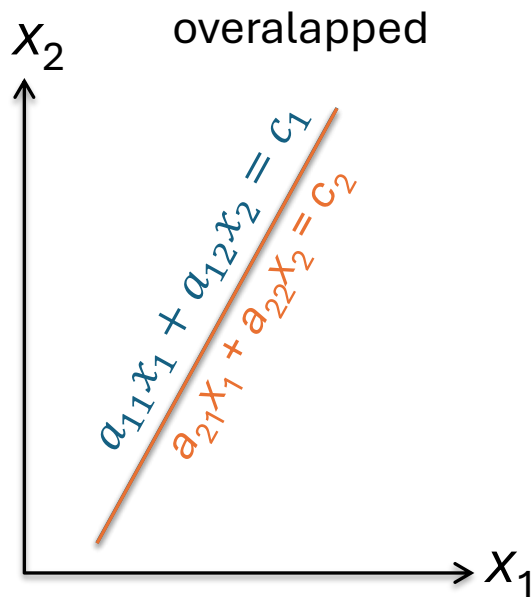
The graphical method



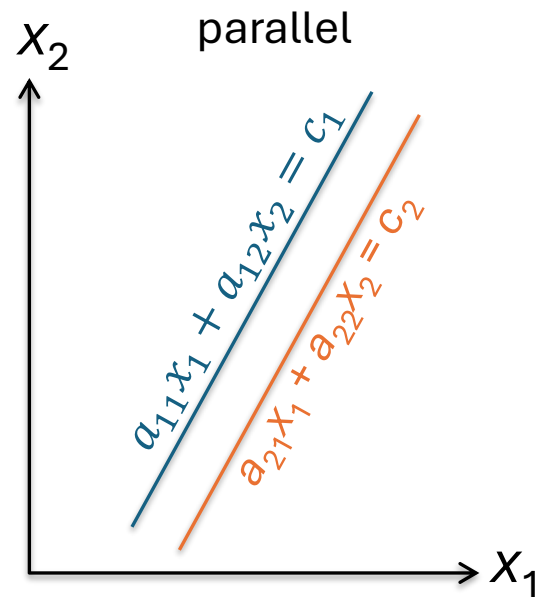
The graphical method



ill-conditioned system:
the slopes are so close that
the point of intersection is
difficult to detect visually



singular system:
infinite solutions



singular system:
no solution

Cramer's rule

- The rule states that each unknown in a system of linear algebraic equations may be expressed as a fraction of two determinants with denominator D (determinant of the equations) and with the numerator obtained from D by replacing the column of coefficient of the unknown in question by the constants $c_1, c_2, \dots, c_n$

- Example

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = c_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = c_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = c_3$$

Cramer's rule

$$[A] = A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$D = \det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$x_1 = \frac{\begin{vmatrix} c_1 & a_{12} & a_{13} \\ c_2 & a_{22} & a_{23} \\ c_3 & a_{32} & a_{33} \end{vmatrix}}{D}$$

$$x_2 = \frac{\begin{vmatrix} a_{11} & c_1 & a_{13} \\ a_{21} & c_2 & a_{23} \\ a_{31} & c_3 & a_{33} \end{vmatrix}}{D}$$

$$x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & c_1 \\ a_{21} & a_{22} & c_2 \\ a_{31} & a_{32} & c_3 \end{vmatrix}}{D}$$

Cramer's rule

- How to find the determinant of a square 2-by-2 matrix?

$$[A] = \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$D = \det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Cramer's rule

- How to find the determinant of a square 3-by-3 matrix?

$$[B] = \mathbf{B} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{aligned} D = \det \mathbf{B} &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \end{aligned}$$

Cramer's rule

- Example

$$\begin{aligned}3x_1 - 0.1x_2 - 0.2x_3 &= 7.85 \\0.1x_1 + 7x_2 - 0.3x_3 &= -19.3 \\0.3x_1 - 0.2x_2 + 10x_3 &= 71.4\end{aligned}$$



$$A \cdot X = C$$

$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0.1 & 7 & -0.3 \\ 0.3 & -0.2 & 10 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.3 \\ 71.4 \end{Bmatrix}$$

$$\det A = 3\{7 \times 10 - (-0.3) \times (-0.2)\} + 0.1\{0.1 \times 10 - (-0.3) \times 0.3\} - 0.2\{0.1 \times (-0.2) - 7 \times 0.3\}$$

$$\det A = 210.353$$

Cramer's rule

$$\mathbf{A}_1 = \begin{bmatrix} 7.85 & -0.1 & -0.2 \\ -19.3 & 7 & -0.3 \\ 71.4 & -0.2 & 10 \end{bmatrix} \quad \mathbf{A}_2 = \begin{bmatrix} 3 & 7.85 & -0.2 \\ 0.1 & -19.3 & -0.3 \\ 0.3 & 71.4 & 10 \end{bmatrix} \quad \mathbf{A}_3 = \begin{bmatrix} 3 & -0.1 & 7.85 \\ 0.1 & 7 & -19.3 \\ 0.3 & -0.2 & 71.4 \end{bmatrix}$$

$$\det \mathbf{A}_1 = |\mathbf{A}_1| = 631.059 \quad \det \mathbf{A}_2 = |\mathbf{A}_2| = -525.8825 \quad \det \mathbf{A}_3 = |\mathbf{A}_3| = 1472.471$$

$$x_1 = \frac{\det \mathbf{A}_1}{\det \mathbf{A}} = \frac{631.058}{210.353} = 3$$

$$x_2 = \frac{\det \mathbf{A}_2}{\det \mathbf{A}} = \frac{-525.8825}{210.353} = -2.5$$

$$x_3 = \frac{\det \mathbf{A}_3}{\det \mathbf{A}} = \frac{1472.471}{210.353} = 7$$

Elimination of unknowns

- The elimination of unknowns by combining equations is an algebraic approach that can be illustrated for a set of two equations

$$\begin{array}{lclclcl} a_{11}x_1 + a_{12}x_2 = c_1 & \Rightarrow & a_{21}(a_{11}x_1 + a_{12}x_2 = c_1) & \Rightarrow & a_{21}a_{11}x_1 + a_{21}a_{12}x_2 = a_{21}c_1 \\ a_{21}x_1 + a_{22}x_2 = c_2 & \Rightarrow & a_{11}(a_{21}x_1 + a_{22}x_2 = c_2) & \Rightarrow & a_{11}a_{21}x_1 + a_{11}a_{22}x_2 = a_{11}c_2 \end{array}$$

$$a_{21}a_{12}x_2 - a_{11}a_{22}x_2 = a_{21}c_1 - a_{11}c_2$$

$$x_2 = \frac{a_{21}c_1 - a_{11}c_2}{a_{21}a_{12} - a_{11}a_{22}}$$

$$x_1 = \frac{c_1 - a_{12}x_2}{a_{11}} = \frac{c_1}{a_{11}} - \frac{a_{12}}{a_{11}} \left(\frac{a_{21}c_1 - a_{11}c_2}{a_{21}a_{12} - a_{11}a_{22}} \right)$$

Elimination of unknowns

- Example

$$3x_1 + 2x_2 = 18 \quad \Rightarrow \quad -(3x_1 + 2x_2 = 18) \quad \Rightarrow \quad -3x_1 - 2x_2 = -18$$

$$-x_1 + 2x_2 = 2 \quad \Rightarrow \quad 3(-x_1 + 2x_2 = 2) \quad \Rightarrow \quad -3x_1 + 6x_2 = 6$$

$$\begin{array}{r} -3x_1 - 2x_2 = -18 \\ -3x_1 + 6x_2 = 6 \\ \hline -8x_2 = -24 \end{array}$$

$$x_2 = \frac{-24}{-8} = 3$$

$$x_1 = \frac{18 - 2x_2}{3} = \frac{18 - 2(3)}{3} = \frac{12}{3} = 4$$

Naïve Gauss elimination

- Strategy
 - Forward elimination of unknowns
 - Backward substitution

- Example

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = c_1 \quad (1)$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = c_2 \quad (2)$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = c_3 \quad (3)$$

Naïve Gauss elimination

- Forward elimination #1
 - eliminate x_1 from eqs. (2) and (3) by using eq. (1) as the pivot

pivot coefficient

pivot equation

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = c_1$$

$$\left(a_{22} - \frac{a_{21}}{a_{11}}a_{12}\right)x_2 + \left(a_{23} - \frac{a_{21}}{a_{11}}a_{13}\right)x_3 = c_2 - \frac{a_{21}}{a_{11}}c_1$$

$$\left(a_{32} - \frac{a_{31}}{a_{11}}a_{12}\right)x_2 + \left(a_{33} - \frac{a_{31}}{a_{11}}a_{13}\right)x_3 = c_3 - \frac{a_{31}}{a_{11}}c_1$$

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = c_1 \quad (1)$$

$$a'_{22}x_2 + a'_{23}x_3 = c'_2 \quad (2')$$

$$a'_{32}x_2 + a'_{33}x_3 = c'_3 \quad (3')$$

Naïve Gauss elimination

- Forward elimination #2
 - eliminate x_2 from eq. (3) by using eq. (2') as the pivot

pivot coefficient

pivot equation

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = c_1$$

$$a'_{22}x_2 + a'_{23}x_3 = c'_2$$

$$\left(a'_{33} - \frac{a'_{32}}{a'_{22}}a'_{23}\right)x_3 = c'_3 - \frac{a'_{32}}{a'_{22}}c'_2$$

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = c_1 \quad (1)$$

$$a'_{22}x_2 + a'_{23}x_3 = c'_2 \quad (2')$$

$$a''_{33}x_3 = c''_3 \quad (3'')$$

Naïve Gauss elimination

- Backward substitution
 - find x_3 from eq. (3''), x_2 from eq. (2'), and x_1 from eq. (1)

$$x_3 = \frac{c_3''}{a_{33}''} \quad \Rightarrow \quad x_2 = \frac{c_2' - a_{23}'x_3}{a_{22}'} \quad \Rightarrow \quad x_1 = \frac{c_1 - a_{12}x_2 - a_{13}x_3}{a_{11}}$$

Naïve Gauss elimination

Forward elimination

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = c_1$$

$$a'_{22}x_2 + a'_{23}x_3 + \cdots + a'_{2n}x_n = c'_2$$

$$a''_{23}x_3 + \cdots + a''_{3n}x_n = c''_3$$

$$\vdots$$
$$\vdots$$
$$\vdots$$

$$a_{nn}^{n-1}x_n = c_n^{n-1}$$

Backward substitution

$$x_n = \frac{c_n^{n-1}}{a_{nn}^{n-1}}$$

$$x_i = \frac{c_i^{i-1} - \sum_{j=i+1}^n a_{ij}^{i-1}x_j}{a_{ii}^{i-1}}, \quad i = n-1, n-2, \dots, 1$$

Naïve Gauss elimination

- Example

$$(1) \quad 3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$

$$(2) \quad 0.1x_1 + 7x_2 - 0.3x_3 = -19.3$$

$$(3) \quad 0.3x_1 - 0.2x_2 + 10x_3 = 71.4$$

Naïve Gauss elimination

■ Forward elimination

- eliminate x_1 from eqs. (2) and (3) by using eq. (1) as the pivot

$$(1) \quad 3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$

$$(2') \quad 0x_1 + 7.0033x_2 - 0.2933x_3 = -19.5617$$

$$(3') \quad 0x_1 - 0.19x_2 + 10.02x_3 = 70.615$$

- eliminate x_2 from eq. (3) by using eq. (2') as the pivot

$$(1) \quad 3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$

$$(2') \quad 0x_1 + 7.0033x_2 - 0.2933x_3 = -19.5617$$

$$(3'') \quad 0x_1 + 0x_2 + 10.0120x_3 = 70.0843$$

Naïve Gauss elimination

- Back substitution

- find x_3 from eq. (3'')

$$x_3 = \frac{70.0843}{10.0120} = 7$$

- substitute x_3 to eq. (2') to get x_2

$$x_2 = \frac{-19.5617 + 0.2933(7)}{7.0033} = -2.5$$

- substitute x_3 and x_2 to eq. (1) to get x_1

$$x_1 = \frac{7.85 + 0.2 \times 7 + 0.1(-2.5)}{3} = 3$$

Naïve Gauss elimination

- Pitfalls of the naïve Gauss elimination
 - it is possible that division by zero can occur during both the forward elimination and the back substitution phases
 - problem of round-off errors can become particularly important when large numbers of equations are to be solved
 - this is due to the fact that every result is dependent on previous results
 - an error in the early steps will tend to propagate, that is, it will cause errors in subsequent steps
 - ill-conditioned systems, where small changes in coefficients result in large changes in the solution

Naïve Gauss elimination

- Techniques for improving solutions
 - determine the largest available coefficient in the column below the pivot element
 - switch the rows so that the largest element is the pivot element



- ➔ rearranging the equations such that the pivot equation is the one that gives the largest pivot coefficient

Gauss-Jordan

- Gauss-Jordan method is a variation of Gauss elimination
 - when an unknown is eliminated in the Gauss-Jordan method, it is eliminated from all other equations rather than just the subsequent ones
 - all rows are normalized by dividing them by their pivot elements
 - the elimination step results in an identity matrix rather than a triangular matrix

- Example

$$(1) \quad 3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$

$$(2) \quad 0.1x_1 + 7x_2 - 0.3x_3 = -19.3$$

$$(3) \quad 0.3x_1 - 0.2x_2 + 10x_3 = 71.4$$

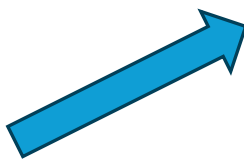
Gauss-Jordan

$$\left[\begin{array}{ccc|c} 3 & -0.1 & -0.2 & 7.85 \\ 0.1 & 7 & -0.3 & -19.3 \\ 0.3 & -0.1 & 10 & 71.4 \end{array} \right]$$



$$\left[\begin{array}{ccc|c} 3/3 & -0.1/3 & -0.2/3 & 7.85/3 \\ 0.1 & 7 & -0.3 & -19.3 \\ 0.3 & -0.1 & 10 & 71.4 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & -0.0333 & -0.0667 & 2.6167 \\ 0.1 & 7 & -0.3 & -19.3 \\ 0.3 & -0.1 & 10 & 71.4 \end{array} \right]$$

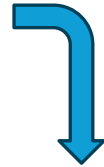


$$\left[\begin{array}{ccc|c} 1 & -0.0333 & -0.0667 & 2.6167 \\ 0 & 7.0033 & -0.2933 & -19.5617 \\ 0 & -0.1900 & 10.0200 & 70.6150 \end{array} \right]$$

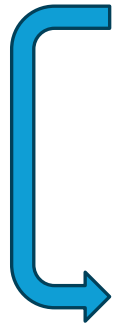


Gauss-Jordan

$$\left[\begin{array}{ccc|c} 1 & -0.0333 & -0.0667 & 2.6167 \\ 0 & 7.0033 & -0.2933 & -19.5617 \\ 0 & -0.1900 & 10.0200 & 70.6150 \end{array} \right]$$



$$\left[\begin{array}{ccc|c} 1 & -0.0333 & -0.0667 & 2.6167 \\ 0/7.0033 & 7.0033/7.0033 & -0.2933/7.0033 & -19.5617/7.0033 \\ 0 & -0.1900 & 10.0200 & 70.6150 \end{array} \right]$$

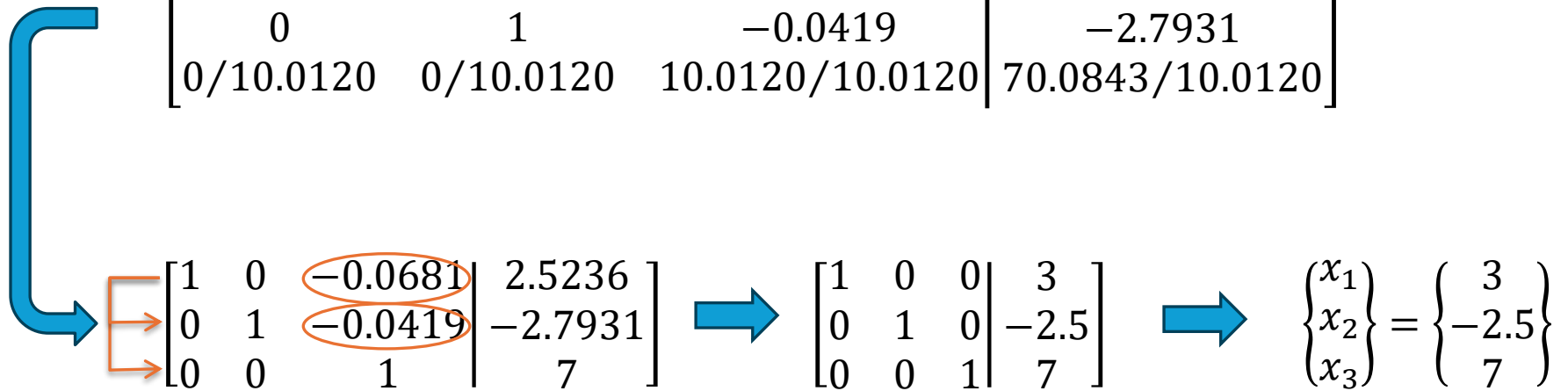


$$\left[\begin{array}{ccc|c} 1 & -0.0333 & -0.0667 & 2.6167 \\ 0 & 1 & -0.0419 & -2.7931 \\ 0 & -0.1900 & 10.0200 & 70.6150 \end{array} \right]$$



$$\left[\begin{array}{ccc|c} 1 & 0 & -0.0681 & 2.5236 \\ 0 & 1 & -0.0419 & -2.7931 \\ 0 & 0 & 10.0120 & 70.0843 \end{array} \right]$$

Gauss-Jordan



$$\begin{bmatrix} 1 & 0 & -0.0681 & 2.5236 \\ 0 & 1 & -0.0419 & -2.7931 \\ 0/10.0120 & 0/10.0120 & 10.0120/10.0120 & 70.0843/10.0120 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -0.0681 & 2.5236 \\ 0 & 1 & -0.0419 & -2.7931 \\ 0 & 0 & 1 & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2.5 \\ 0 & 0 & 1 & 7 \end{bmatrix} \rightarrow \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 3 \\ -2.5 \\ 7 \end{Bmatrix}$$

Gauss-Jordan

- It has more operations than Gauss elimination by 50%
- It has the same pitfalls, i.e.
 - division by zero
 - large round-off errors

LU decomposition

- Gauss elimination $\rightarrow$ LU decomposition

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{[A]} \rightarrow \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ f_{21} & 1 & 0 \\ f_{31} & f_{32} & 1 \end{bmatrix}}_{\{L\}} \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix}}_{[U]}$$

$$f_{21} = \frac{a_{21}}{a_{11}} \quad f_{31} = \frac{a_{31}}{a_{11}} \quad f_{32} = \frac{a'_{32}}{a'_{22}}$$

***LU* decomposition**

$$[L][U] = [A]$$

$$[L]\{D\} = \{C\}$$

$$[U]\{X\} = \{D\}$$

LU decomposition

- Steps to obtain U following Gauss elimination
 - Step 1: forward elimination

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & a'_{32} & a'_{33} \end{bmatrix}$$

- Step 2: forward elimination

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & a'_{32} & a'_{33} \end{bmatrix} \rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix}$$

$[U]$

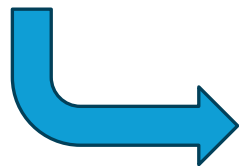
LU decomposition

- Example

$$(1) \quad 3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$

$$(2) \quad 0.1x_1 + 7x_2 - 0.3x_3 = -19.3$$

$$(3) \quad 0.3x_1 - 0.2x_2 + 10x_3 = 71.4$$



$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0.1 & 7 & -0.3 \\ 0.3 & -0.2 & 10 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.3 \\ 71.4 \end{Bmatrix}$$

$[A] \qquad \qquad \{X\} \qquad \qquad \{C\}$

LU decomposition

$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0.1 & 7 & -0.3 \\ 0.3 & -0.2 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & -0.1 & -0.2 \\ 0 & 7.0033 & -0.2933 \\ 0 & -0.19 & 10.0200 \end{bmatrix} \rightarrow [U] = \begin{bmatrix} 3 & -0.1 & -0.2 \\ 0 & 7.0033 & -0.2933 \\ 0 & 0 & 10.0120 \end{bmatrix}$$

$$f_{21} = \frac{a_{21}}{a_{11}} = \frac{0.1}{3} = 0.0333$$

$$f_{31} = \frac{a_{31}}{a_{11}} = \frac{0.3}{3} = 0.1$$

$$f_{32} = \frac{a'_{32}}{a'_{22}} = \frac{-0.19}{7.0033} = -0.0271$$



$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ 0.0333 & 1 & 0 \\ 0.1000 & -0.0271 & 1 \end{bmatrix}$$

LU decomposition

$$[L]\{D\} = \{C\}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.0333 & 1 & 0 \\ 0.1000 & 0.0271 & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.3 \\ 71.4 \end{Bmatrix}$$

$$d_1 = 7.85$$

$$0.0333d_1 + d_2 = -19.3 \Leftrightarrow d_2 = -19.3 - 0.0333(7.85) = -19.5617$$

$$0.1d_1 + 0.0271d_2 + d_3 = 71.4$$

$$\Leftrightarrow d_3 = 71.4 - 0.1(7.85) - 0.0271(-19.5617) = 70.0843$$

LU decomposition

$$[U]\{X\} = \{D\}$$

$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0 & 7.0033 & -0.2933 \\ 0 & 0 & 10.0120 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.5617 \\ 70.0843 \end{Bmatrix}$$

$$10.0120x_3 = 70.0843 \Leftrightarrow x_3 = \frac{70.0843}{10.0120} = 7$$

$$7.0033x_2 - 0.2933x_3 = -19.5617 \Leftrightarrow x_2 = \frac{-19.5617 + 0.2933(7)}{7.0033} = -2.5$$

$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$

$$\Leftrightarrow x_1 = \frac{7.85 + 0.1(-2.5) + 0.2(7)}{3} = 3$$


$$\{X\} = \begin{Bmatrix} 3 \\ -2.5 \\ 7 \end{Bmatrix}$$

The matrix inverse

$$[A] \cdot \{X\} = \{C\} \Rightarrow \{X\} = [A]^{-1} \cdot \{C\}$$

$$\left[\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array} \right] \quad \rightarrow \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & a_{11}^{-1} & a_{12}^{-1} & a_{13}^{-1} \\ 0 & 1 & 0 & a_{21}^{-1} & a_{22}^{-1} & a_{23}^{-1} \\ 0 & 0 & 1 & a_{31}^{-1} & a_{32}^{-1} & a_{33}^{-1} \end{array} \right]$$

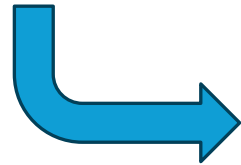
The matrix inverse

- Example

$$(1) \quad 3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$

$$(2) \quad 0.1x_1 + 7x_2 - 0.3x_3 = -19.3$$

$$(3) \quad 0.3x_1 - 0.2x_2 + 10x_3 = 71.4$$



$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0.1 & 7 & -0.3 \\ 0.3 & -0.2 & 10 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.3 \\ 71.4 \end{Bmatrix}$$

$[A] \qquad \qquad \{X\} \qquad \qquad \{C\}$

The matrix inverse

$$[A] = \left[\begin{array}{ccc|ccc} 3 & -0.1 & -0.2 & 1 & 0 & 0 \\ 0.1 & 7 & -0.3 & 0 & 1 & 0 \\ 0.3 & -0.2 & 10 & 0 & 0 & 1 \end{array} \right] \rightarrow [A] = \left[\begin{array}{ccc|ccc} 1 & -0.0333 & -0.0667 & 0.3333 & 0 & 0 \\ 0.1 & 7 & -0.3 & 0 & 1 & 0 \\ 0.3 & -0.2 & 10 & 0 & 0 & 1 \end{array} \right]$$



$$[A] = \left[\begin{array}{ccc|ccc} 1 & -0.0333 & -0.0667 & 0.3333 & 0 & 0 \\ 0 & 7.0033 & -0.2933 & -0.0333 & 1 & 0 \\ 0 & -0.1900 & 10.0200 & -0.0999 & 0 & 1 \end{array} \right]$$

The matrix inverse

$$[A] = \left[\begin{array}{ccc|ccc} 1 & -0.0333 & -0.0667 & 0.3333 & 0 & 0 \\ 0 & 1 & -0.0417 & -0.0047 & 0.1422 & 0 \\ 0 & -0.1900 & 10.0200 & -0.0999 & 0 & 1 \end{array} \right]$$



$$[A] = \left[\begin{array}{ccc|ccc} 1 & 0 & -0.0681 & 0.3318 & 0.0047 & 0 \\ 0 & 1 & -0.0417 & -0.0047 & 0.1422 & 0 \\ 0 & 0 & 10.0121 & -0.1009 & 0.0270 & 1 \end{array} \right]$$

The matrix inverse

$$[A] = \left[\begin{array}{ccc|ccc} 1 & 0 & -0.0681 & 0.3318 & 0.0047 & 0 \\ 0 & 1 & -0.0417 & -0.0047 & 0.1422 & 0 \\ 0 & 0 & 1 & -0.0101 & 0.0027 & 0.0999 \end{array} \right]$$



$$[A] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.3325 & 0.0049 & 0.0068 \\ 0 & 1 & 0 & -0.0052 & 0.1423 & 0.0042 \\ 0 & 0 & 1 & -0.0101 & 0.0027 & 0.0999 \end{array} \right]$$

$$[A]^{-1}$$

The matrix inverse

$$\{X\} = [A]^{-1}\{C\}$$

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} 0.3325 & 0.0049 & 0.0068 \\ -0.0052 & 0.1423 & 0.0042 \\ -0.0101 & 0.0027 & 0.0999 \end{bmatrix} \begin{Bmatrix} 7.85 \\ -19.3 \\ 71.4 \end{Bmatrix}$$

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 3.0004 \\ -2.4881 \\ 7.0002 \end{Bmatrix}$$

The matrix inverse

- Pitfalls
 - The matrix inverse may not exist
- MS Excel
 - =MINVERSE() cntrl+shift+enter
 - =MMULT() cntrl+shift+enter

Jacobi

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = c_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = c_2$$

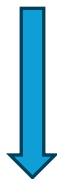
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = c_3$$

$$x_1 = \frac{c_1 - a_{12}x_2 - a_{13}x_3}{a_{11}}$$

$$x_2 = \frac{c_2 - a_{21}x_1 - a_{23}x_3}{a_{22}}$$

$$x_3 = \frac{c_3 - a_{31}x_1 - a_{32}x_2}{a_{33}}$$

initial guesses, $x_1^0 = 0$
 e.g. $x_i^0 = 0$ $x_2^0 = 0$
 $x_3^0 = 0$



$$x_1^1 = \frac{c_1 - a_{12}x_2^0 - a_{13}x_3^0}{a_{11}}$$

$$x_2^1 = \frac{c_2 - a_{21}x_1^0 - a_{23}x_3^0}{a_{22}}$$

$$x_3^1 = \frac{c_3 - a_{31}x_1^0 - a_{32}x_2^0}{a_{33}}$$



$$x_1^{n+1} = \frac{c_1 - a_{12}x_2^n - a_{13}x_3^n}{a_{11}}$$

$$x_2^{n+1} = \frac{c_2 - a_{21}x_1^n - a_{23}x_3^n}{a_{22}}$$

$$x_3^{n+1} = \frac{c_3 - a_{31}x_1^n - a_{32}x_2^n}{a_{33}}$$

continue the
 iteration until
 $x_i^{n+1} \approx x_i^n, \forall x_i$

Jacobi

- Example

$$(1) \quad 3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$

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Jacobi

n	x_1^n	x_2^n	x_3^n	$ \Delta \mathbf{x} _{max}$
0	0	0	0	
1	2.616667	-2.75714	7.14	7.14
2	3.000762	-2.48852	7.006357	0.3840952
3	3.000806	-2.49974	7.000207	0.0112146
4	3.000022	-2.5	6.999981	0.0007839
5	2.999999	-2.5	6.999999	2.385E-05

Gauss-Seidel

$$x_1^1 = \frac{c_1 - a_{12}x_2^0 - a_{13}x_3^0}{a_{11}}$$

$$x_2^1 = \frac{c_2 - a_{21}x_1^0 - a_{23}x_3^0}{a_{22}}$$

$$x_3^1 = \frac{c_1 - a_{31}x_1^0 - a_{32}x_2^0}{a_{33}}$$



$$x_1^{n+1} = \frac{c_1 - a_{12}x_2^n - a_{13}x_3^n}{a_{11}}$$

$$x_2^{n+1} = \frac{c_2 - a_{21}x_1^{n+1} - a_{23}x_3^n}{a_{22}}$$

$$x_3^{n+1} = \frac{c_1 - a_{31}x_1^{n+1} - a_{32}x_2^{n+1}}{a_{33}}$$

continue the
iteration until
 $x_i^{n+1} \approx x_i^n, \forall x_i$

Gauss-Seidel

- Example

$$(1) \quad 3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$

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Gauss-Seidel

n	x_1^n	x_2^n	x_3^n	$ \Delta x _{max}$
0	0	0	0	
1	2.616667	-2.79452	7.00561	7.0056095
2	2.990557	-2.49962	7.000291	0.3738898
3	3.000032	-2.49999	6.999999	0.0094754
4	3	-2.5	7	3.155E-05
5	3	-2.5	7	3.544E-07

Successive over-relaxation

$$x_1^{n+1} = \frac{c_1 - a_{12}x_2^n - a_{13}x_3^n}{a_{11}}$$

$$x_2^{n+1} = \frac{c_2 - a_{21}[\lambda x_1^{n+1} + (1 - \lambda)x_1^n] - a_{23}x_3^n}{a_{22}}$$

$$x_3^{n+1} = \frac{c_3 - a_{31}[\lambda x_1^{n+1} + (1 - \lambda)x_1^n] - a_{32}[\lambda x_2^{n+1} + (1 - \lambda)x_2^n]}{a_{33}}$$

$$\lambda x_i^{n+1} + (1 - \lambda)x_i^n \begin{cases} 0 < \lambda < 1 & \text{under-relaxation} \\ 1 < \lambda < 2 & \text{over-relaxation} \end{cases}$$

Successive over-/under-relaxation

- Example

$$(1) \quad 3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$

$$(2) \quad 0.1x_1 + 7x_2 - 0.3x_3 = -19.3$$

$$(3) \quad 0.3x_1 - 0.2x_2 + 10x_3 = 71.4$$

Successive over-/under-relaxation

$$\lambda = 1.5$$

n	x_1^n	x_2^n	x_3^n	$ \Delta \mathbf{x} _{max}$
0	0	0	0	
1	2.616667	-2.81321	6.937854	6.93785357
2	2.985416	-2.50509	6.997886	0.36874976
3	2.999689	-2.50019	6.99984	0.01427299
4	2.999983	-2.50001	6.999998	0.00029368
5	3	-2.5	7	1.6465E-05



Numerical Methods

Linear Algebraic Equations