Statistika Teknik

Distribusi Probabilitas Kontinu

Distribusi Probabilitas Kontinu

- Distribusi normal
- Distribusi seragam (uniform distributions)
- Distribusi eksponensial
- Distribusi gamma
- Distribusi log normal
- Distribusi nilai ekstrem (*extreme value distributions*)
    - *Extreme Value Type I*
    - *Extreme Value Type III Minimum (Weibull)*
- Distribusi beta
- Distribusi Pearson

Distribusi Normal

➡ ST06 Distribusi Normal

Distribusi Seragam

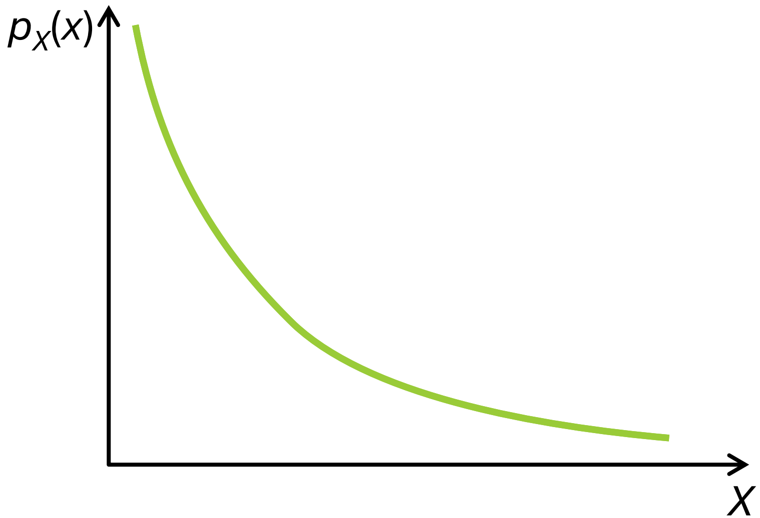
pdf $$p_X(X) = \frac{1}{\beta - \alpha}, \qquad \alpha \le x \le \beta$$

cdf $$P_X(X) = \frac{x - \alpha}{\beta - \alpha}, \qquad \alpha \le x \le \beta$$

rerata $$\mathrm{E}(X) = \frac{1}{2}(\beta + \alpha)$$

varians $$\mathrm{var}(X) = \frac{1}{12}(\beta - \alpha)^2$$

Distribusi Eksponensial



Skewness coefficient:

$c_s = 2$ konstan

Parameter λ:

$\hat{\lambda} = \frac{1}{\bar{\bar{X}}}$ estimasi

pdf $p_X(X) = \lambda e^{-\lambda x}$ $x > 0, \lambda > 0$

cdf $P_X(X) = \int_0^x \lambda e^{-\lambda t} dt = 1 - e^{-\lambda x}$ $x > 0$

Distribusi Gamma

- Penjumlahan sejumlah *n* variabel random berdistribusi eksponensial, masing-masing berparameter λ, menghasilkan variabel random berdistribusi gamma dengan parameter λ.

pdf $$p_X(X) = \frac{\lambda^\eta x^{\eta-1}\; e^{-\lambda x}}{\Gamma(\eta)} \qquad x, \lambda, \eta > 0$$

cdf $$P_X(X) = \int_0^x \frac{\lambda^\eta x^{\eta-1}\; e^{-\lambda t}}{\Gamma(\eta)} dt$$

$$P_X(X) = 1 - \; e^{-\lambda x} \sum_{i=0}^{\eta-1} \frac{(\lambda x)^i}{i!} \qquad \eta = \text{integer}$$

Distribusi Gamma

$\Gamma(\eta)$ = fungsi gamma

$\Gamma(\eta) = (\eta - 1)!$ $\quad \eta = 1,2,3, \ldots$

$\Gamma(\eta + 1) = \eta\Gamma(\eta)$ $\quad \eta > 0$

$$\Gamma(\eta) = \int_0^\infty t^{\eta-1} e^{-\eta} dt \quad \eta > 0$$

$\Gamma(1) = \Gamma(2) = 1$

$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Mean:

$$\mathrm{E}(X) = \frac{\eta}{\lambda}$$

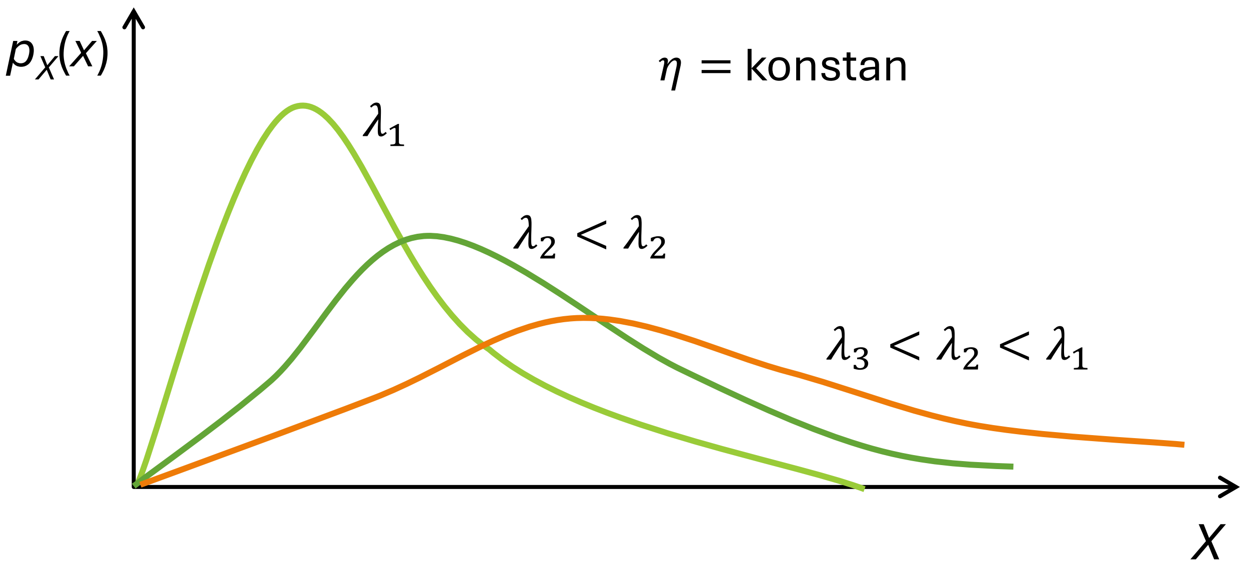
Variance:

$$\mathrm{var}(X) = \frac{\eta}{\lambda^2}$$

Skewness coefficient:

$$\gamma = \frac{2}{\sqrt{\eta}}$$

Distribusi Gamma



$$p_X(x) = \frac{1}{\Gamma(\eta)} \lambda^{\eta} x^{\eta-1} e^{-\lambda x}, \qquad x, \lambda, \eta > 0$$

Distribusi Log Normal

- Variabel random X
  - Jika disusun dari penjumlahan sejumlah pengaruh variabel kecil, maka X kemungkinan besar berdistribusi normal.
  - Jika disusun dari perkalian sejumlah pengaruh variabel kecil, maka ln X kemungkinan besar berdistribusi normal.

$$X = X_1 + X_2 + \cdots + X_n$$

→ X_i berdistribusi normal

X berdistribusi normal

$$X = X_1 \cdot X_2 \cdot \ldots \cdot X_n$$

$$\ln X = \ln X_1 + \ln X_2 + \cdots + \ln X_n$$

→ X_i berdistribusi normal

ln X berdistribusi normal

Distribusi Log Normal

$Y = \ln X$

$Y_i = \ln X_i$ $\quad Y = Y_1 + Y_2 + \cdots + Y_n$ ➡ berdistribusi normal

$$p_Y(y) = \frac{1}{\sigma_Y\sqrt{2\pi}} e^{-\frac{1}{2}(Y-\mu_Y)^2/\sigma_Y{}^2}, \qquad -\infty < y < +\infty$$

Distribusi *X*?

$$p_X(x) = p_Y(y)\left|\frac{dy}{dx}\right| \qquad Y = \ln X \Rightarrow \left|\frac{dy}{dx}\right| = \frac{1}{x}, \qquad x > 0$$

$$p_X(x) = \frac{1}{x\sigma_Y\sqrt{2\pi}} e^{-\frac{1}{2}(\ln X-\mu_Y)^2/\sigma_Y{}^2}, \qquad x > 0$$

Distribusi Log Normal

Estimasi μ_Y dan σ_Y

$$\left.\begin{aligned}\mu_Y &\longrightarrow \bar{Y}\\ \sigma_Y &\longrightarrow s_Y\end{aligned}\right\}$$ Data x_i ditransformasikan dulu menjadi $y_i = \ln x_i$

Cara lain:

$$\bar{Y} = \frac{1}{2}\ln\left(\frac{\overline{X^2}}{{c_v}^2 + 1}\right)$$

$$s_Y{}^2 = \ln\left({c_v}^2 + 1\right)$$

$$c_v = \frac{s_X}{\bar{X}}$$

c_v koefisien variansi data asli

Distribusi Log Normal

- Rerata

$$\mathrm{E}(X) = e^{\left(\mu_Y + \frac{1}{2}{\sigma_Y}^2\right)}$$

- Varians

$$\mathrm{var}(X) = {\mu_X}^2 \left(e^{{\sigma_Y}^2} - 1\right)$$

- Koefisien variasi

$$c_v = \left(e^{{\sigma_Y}^2} - 1\right)^2$$

- *Coefficient of skew*

$$\gamma = 3c_v + {c_v}^3$$

Distribusi Nilai Ekstrem

- Contoh nilai ekstrem
  - Debit banjir
  - Debit minimum
- Nilai-nilai ekstrem variabel random juga merupakan variabel random.
- Distribusi variabel random nilai ekstrem tsb bergantung pada:
  - distribusi variabel random tempat asal variabel nilai ekstrem tsb diperoleh → *parent distribution*
  - jumlah/ukuran sampel

Distribusi Nilai Ekstrem

- Contoh
  - Variabel random

    $$X = X_1, X_2, \ldots, X_n$$

  - Y nilai ekstrem variabel random tersebut

    $$P_Y(y) = \text{prob}(Y \leq y)$$

    $$P_{X_i}(x) = \text{prob}(X_i \leq x)$$

    $P_Y(y)$ = prob($Y \leq y$) = prob(semua Y yang lebih kecil daripada y)

Distribusi Nilai Ekstrem

maka:

$$P_Y(y) = P_{X_1}(y) \cdot P_{X_1}(y) \cdot \ldots \cdot P_{X_1}(y)$$

$$= [P_X(y)]^n$$

$$P_Y(y) = \frac{dP_Y(y)}{dy}$$

$$= n[P_X(y)]^{n-1} \frac{dP_X(y)}{dy}$$

$$= n[P_X(y)]^{n-1} p_X(y)$$

Distribusi Nilai Ekstrem

- Contoh
  - Waktu antara 2 hujan berurutan berdistribusi eksponensial.
  - Waktu rata-rata antara 2 hujan = 4 hari
  - Waktu antara tsb merupakan kejadian *independent* satu dengan yang lain
  - Dicari:
    - waktu antara terbesar, misal probabilitas waktu antara tsb lebih besar daripada 8 hari.

Distribusi Nilai Ekstrem

- Ditinjau 10 kejadian hujan

h *h* *h* *h* *h* *h* *h* *h* *h* *h*

x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9 → $n = 9$

Distribusi Eksponensial:

$$p_X(x) = \lambda e^{-\lambda x}, \qquad x > 0, \lambda > 0$$

$$P_X(x) = 1 - e^{-\lambda x}, \qquad x > 0$$

$$\mathrm{E}(X) = \lambda^{-1} \Longrightarrow \lambda = \frac{1}{\mathrm{E}(X)} \qquad \hat{\lambda} = \frac{1}{\bar{\bar{X}}}$$

Distribusi Nilai Ekstrem

$$p_X(x) = \frac{1}{4} e^{-\frac{1}{4}x}$$

$$P_X(x) = 1 - e^{-\frac{1}{4}x}$$

$$\begin{aligned} P_Y(8) &= \text{prob}(Y \le 8) = \text{prob}(\text{semua } x \le 8) \\ &= \left[P_X(8)\right]^9 \\ &= \left(1 - e^{-2}\right)^9 \\ &= 0.271 \end{aligned}$$

$$\begin{aligned} \text{prob}(Y > 8) &= 1 - 0.271 \\ &= 0.729 \end{aligned}$$

Distribusi Nilai Ekstrem

- Permasalahan yang sering ditemui adalah bahwa jenis *parent distribution* tidak diketahui.
- Hal ini diatasi dengan
  - ukuran sampel cukup besar, *n* >>
  - pemakaian distribusi asimtotis
  - dikenal tiga jenis distribusi asimtotis
    - *Type* I – *parent distribution unbounded in direction of the desired extreme and all moments of the distribution exist (exponential type distributions)*
    - *Type* II – *parent distribution unbounded in direction of the desired extreme and all moments of the distribution do not exist (Cauchy type distributions)*
    - *Type* III – *parent distribution bounded in the direction of the desired extreme (limited distributions)*

Distribusi Nilai Ekstrem

- Permasalahan yang menjadi interest umumnya menyangkut nilai-nilai ekstrem maximum atau extrem minimum.
- Beberapa contoh parent distributions
  - *Type* I – extreme value largest – normal, lognormal, eksponential, gamma
  - *Type* I – *extreme value smallest* – normal
  - *Type* II – *extreme value largest or smallest* – distribusi Cauchy
  - *Type* III – *extreme value largest* – distribusi beta
  - *Type* III – *extreme value smallest* – beta, lognormal, gamma, eksponential

Distribusi Nilai Ekstrem

- Permasalahan di bidang hidrologi
  - Type II – *extreme value largest or smallest*
    - jarang dijumpai/dipakai
  - Type I – *extreme value largest*
    - nilai ekstrem maksimum sering mengikuti distribusi jenis ini mengingat banyak variabel hidrologi *unbounded* di sisi kanan
  - Type III – *extreme value smallest*
    - nilai ekstrem minimum sering mengikuti distribusi jenis ini mengingat banyak variabel hidrologi *bounded* di sisi kiri oleh nilai nol

Type I Extreme Value Distribution (Distribusi Gumbel)

$$p_X(x) = \frac{\exp\{\mp(x-\beta)/\alpha - \exp[\mp(x-\beta)/\alpha]\}}{\alpha}$$

$$-\infty < x < +\infty$$
$$-\infty < \beta < +\infty$$
$$\alpha > 0$$

− untuk nilai maksimum

\+ untuk nilai minimum

α = skala

β = lokasi = *mode*

Type I Extreme Value Distribution (Distribusi Gumbel)

$$\mathrm{E}(X) = \beta + 0.577\alpha \quad \text{(maksimum)}$$

$$= \beta - 0.577\alpha \quad \text{(minimum)}$$

$$\mathrm{var}(X) = 1.645\alpha^2 \quad \text{(maksimum/minimum)}$$

$$\gamma = 1.1396 \quad \text{(maksimum)}$$

$$= -1.1396 \quad \text{(minimum)}$$

Type I Extreme Value Distribution (Distribusi Gumbel)

- Dengan memakai transformasi

$$p_Y(y) = \exp[\mp y - \exp(\mp y)] \qquad Y = \frac{x - \beta}{\alpha}$$

$-$ untuk nilai maksimum

$+$ untuk nilai minimum

$$P_Y(y) = \int_{-\infty}^{+\infty} \exp[\mp t - \exp(\mp t)]\, dt \qquad -\infty < y < +\infty$$

$$= \exp[-\exp(-y)] \qquad \text{(maksimum)}$$

$$= 1 - \exp[-\exp(y)] \qquad \text{(minimum)}$$

$$P_{\min}(y) = 1 - P_{\max}(-y)$$

Type I Extreme Value Distribution (Distribusi Gumbel)

- Estimasi parameter α dan β

$$\hat{\alpha} = \frac{s}{1.283}$$

$$\hat{\beta} = \bar{X} - 0.45s \qquad \text{(maksimum)}$$

$$= \bar{X} + 0.45s \qquad \text{(minimum)}$$

Type II Extreme Value Distribution

$$p_X(x) = 0 \quad \text{jika } x \le \beta$$

$$= \exp\left[-\left(\frac{u-\beta}{x-\beta}\right)^k\right] \quad \text{jika } x \ge \beta$$

$k > 0 \rightarrow$ bentuk

$u - \beta > 0 \rightarrow$ skala

$\beta \rightarrow$ lokasi

$$\mathrm{E}(X) = \beta + (u-\beta)\Gamma\left(1-\frac{1}{k}\right)$$

$$\mathrm{var}(X) = (u-\beta)^2\left[\Gamma\left(1-\frac{2}{k}\right) - \Gamma^2\left(1-\frac{1}{k}\right)\right] \qquad k > 2$$

Type III Extreme (Minimum) Value Distribution (Distribusi Weibull)

$$p_X(x) = \alpha x^{\alpha-1}\beta^{-\alpha}\exp\left[-\left(\frac{x}{\beta}\right)^2\right] \quad x \geq 0$$

$$P_X(x) = 1 - \exp\left[-\left(\frac{x}{\beta}\right)^2\right]$$

$$\mathrm{E}(X) = \beta\Gamma\left(1+\frac{1}{\alpha}\right) \qquad \mathrm{var}(X) = \beta^2\left[\Gamma\left(1+\frac{2}{\alpha}\right) - \Gamma^2\left(1+\frac{1}{\alpha}\right)\right]$$

$$\gamma = \frac{\Gamma\left(1+\frac{3}{\alpha}\right) - 3\Gamma\left(1+\frac{2}{\alpha}\right)\Gamma\left(1+\frac{1}{\alpha}\right) + 2\Gamma^3\left(1+\frac{1}{\alpha}\right)}{\left[\Gamma\left(1+\frac{2}{\alpha}\right) - \Gamma^2\left(1+\frac{1}{\alpha}\right)\right]^{3/2}}$$

Type III Extreme (Minimum) Value Distribution (Distribusi Weibull)

Estimates:

$$\lambda = \beta^{-\alpha}$$

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} {x_i}^{\hat{\alpha}}}$$

$$\hat{\alpha} = \frac{n}{\hat{\lambda} \sum_{i=1}^{n} {x_i}^{\hat{\alpha}} \ln x_i - \sum_{i=1}^{n} \ln x_i}$$

$$\hat{\beta} = (\hat{\alpha})^{-1/\hat{\alpha}}$$

Extreme Value Distributions

- Silakan baca diskusi pada hlm. 118 (Haan, 1982)

Distribusi Beta

- Distribusi yang memiliki batas atas dan batas bawah

$$p_X(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{\mathrm{B}(\alpha,\beta)}, \qquad 0 < x < 1, \qquad \alpha,\beta > 0$$

Beta function: $$\mathrm{B}(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx$$

$$= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

$$\mathrm{E}(X) = \frac{\alpha}{\alpha+\beta}$$

$$\mathrm{var}(X) = \frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$$

Pearson Type III Distribution

$$p_X(x) = p_0 \left(1 + \frac{x}{\alpha}\right)^{\alpha/5} e^{-x/\delta}$$

- mode di $x = 0$
- batas bawah di $x = -\alpha$

- Dengan transformasi (translasi) sehingga:
  - mode di $x = \alpha$
  - batas bawah di $x = 0$

$$p_X(x) = p_0 e^{-(x-\alpha)/\delta} \left(\frac{x}{\alpha}\right)^{\alpha/\beta}$$

Distribusi Probabilitas Kontinu

Distribusi chi-kuadrat
Distribusi t
Distribusi F

Distribusi Chi-kuadrat

$$Z = \frac{X - \mu}{\sigma}$$ variabel random berdistribusi normal

$$Y = \sum_{i=1}^{n} {z_i}^2$$ berdistribusi chi-kuadrat dengan *n degrees of freedom*

Distribusi chi-kuadrat = distribusi gamma dengan $\lambda = {}^1/_2$ dan $\eta =$ kelipatan ${}^1/_2$

$$p_{\chi^2}(x) = \frac{x^{-(1-\nu/2)} e^{-x/2}}{2^{\nu/2}\Gamma(\nu/2)} \quad x, \nu > 0, \nu = 2\eta$$

$$\mathrm{E}\left(\chi^2\right) = \nu \qquad \mathrm{var}\left(\chi^2\right) = 2\nu, \qquad \hat{\nu} = \bar{X}$$

Distribusi *t*

Y = normal baku
U = chi-kuadrat

Y dan *U* *independent*

$X = Y\dfrac{\sqrt{\nu}}{\sqrt{U}}$ → berdistribusi *t* dengan v *degrees of freedom*

$$p_T(t) = \frac{\Gamma\left[\frac{1}{2}(\nu+1)\right]\left(1+t^2/n\right)^{-\frac{1}{2}(\nu+1)}}{\sqrt{\pi\nu}\,\Gamma(\nu/2)}, \qquad -\infty < t < +\infty, \nu > 0$$

$$\mathrm{E}(T) = 0 \qquad \mathrm{var}(T) = \frac{\nu}{\nu-2}\ \text{untuk}\ \nu > 2$$

Distribusi *F*

U = chi-square dengan γ = *m degrees of freedom*
V = chi-square dengan γ = *n degrees of freedom*
} *U* dan *V independent*

maka:

$$X = \left(\frac{U}{m}\right)\left(\frac{V}{n}\right)$$

→ berdistribusi F dengan $\gamma_1 = m$ dan $\gamma_2 = n$ *degrees of freedom*

$$p_F(f) = \frac{\Gamma\left[\frac{1}{2}(\gamma_1 + \gamma_2)\gamma_1^{\gamma_1/2}\gamma_2^{\gamma_2/2} f^{\frac{1}{2}(\gamma_1 - 2)}\right]}{(\gamma_2 + \gamma_1 f)^{\frac{1}{2}(\gamma_1 + \gamma_2)}\Gamma(\gamma_1/2)\Gamma(\gamma_2/2)}, \qquad \gamma_1, \gamma_2 > 0$$

$$\mathrm{E}(F) = \frac{\gamma_1}{\gamma_2 - 2} \qquad \mathrm{var}(F) = \frac{{\gamma_2}^2(\gamma_1 + 2)}{\gamma_1(\gamma_2 - 2)(\gamma_2 - 4)}$$

Terima kasih